A Multilevel Embedded Finite Element Method for the Modeling of Crack Branching

Arun Raina

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Institute of Applied Mechanics (CE) · Chair I · Micromechanics of Materials Group
University of Stuttgart, 70550 Stuttgart, Pfaffenwaldring 7, Germany

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Abstract

Theoretical foundations of dynamic fracture highly deviate from the experimental results under higher rate of external loadings. In addition to certain microstructural changes in the localization zone, complex phenomena like micro and macro branching may occur in case of brittle materials. Advanced finite element (FE) techniques like the ‘embedded finite element method’ can model failure of solids in a computationally efficient manner. Failure in the form of propagating cracks or shearbands appear as jumps in the resulting displacement field also called as ‘strong discontinuities (SDs)’. In the embedded finite element method, the strong discontinuity is embedded into a finite element by enhancing its strain field in a discrete physically motivated manner. Such an approach is made possible by identifying new deformation modes of a FE with a discontinuity as a constant and a linear separation mode where the latter allows for a locking free formulation in quadrilateral plane elements (Linder and Armero, 2007). The key aspect of the embedded finite element method is the local nature of enhancement allowing for the static condensation of the local parameters introduced in the definition of the displacement jumps. For the higher rates of the dynamic loadings, a propagating crack may split into multiple crack surfaces as seen in experimental results of Sharon et al. (1996). Such a phenomena of crack branching cannot be captured by a FE with a single discontinuity. For a precise simulation of the state of the material at the branching crack tip, multiple SDs should be allowed to develop in a single FE. Contrary to the work in Raina and Linder (2010), this is achieved by the application of the embedded finite element method at multiple levels in a new methodology called as multilevel embedded finite element method. The FE at the crack tip is now treated as a new boundary value problem subjected to certain kinematic constraints on its boundary and is adaptively discretized into a number of sub-elements. Coupling between quantities at different levels of discretization is obtained by matching their virtual energies. Finally, the results obtained by the numerical simulations are compared with the experimental results of dynamic fracture performed in Sharon et al. (1996).

1 Introduction

The failure of solids is often caused by excessive loading which exceeds the tensile strength of the solid. At the atomistic scale this can be explained by breaking of bonds between the atoms due to tensile stresses which are greater than the binding forces between the atoms. At the macroscopic level, failure is evident in the form of cracks in case of brittle materials or shear bands in case of ductile metals with the initially smooth distribution of strain changing into a highly localized one at the onset of failure. This strain localization is accompanied by the softening of the material before the final failure. For a given solid, however, the failure does not always happen in the same way regardless of the conditions. The loading rate plays a critical role in the behavior of fracture e.g. for a very high loading rate (i.e. high rate of strain), a single propagating crack may branch into multiple cracks.

1.1 Background

The strain localization cannot be captured by the conventional finite element theories as the underlying numerical problem becomes ill-posed in the softening regime of the stress-strain relation. Various numerical techniques exist which attempt to capture the strain localization. One such technique is based on the smeared-crack models seen in Rots et al. (1985) or Oliver (1989) which lead to a spurious stress transfer across an open crack. Another approach of discrete-crack models applied in Ortiz and Pandolfi (1999), Xu and Needleman (1994), Zhang et al. (2007), Zhou and Molinari (2004) to name a few references, with special interfaces between conventional elements requires frequent re-meshing in order to allow for crack propagation in the correct direction. Other techniques rely on nodally-based enrichments, like the so-called extended finite element method developed in Belytschko and Black (1999), Dolbow et al. (2000), Moës et al.
(1999), Wells and Sluys (2001), Wells et al. (2002) and many others, also referred to as XFEM, which capture localization by increasing the total number of global degrees of freedom. The restrictions of the aforementioned techniques lead to the development of a new method that allows efficient modeling of regions with highly localized strains by introducing jumps in the displacement field also called as ‘strong discontinuities’. The initial development of this method can be traced to the work of Dvorkin et al. (1990) and the general framework for finite elements with embedded strong discontinuities presented in Simo et al. (1993), Armero and Garikipati (1995, 1996), Oliver (1996), Larsson et al. (1998) and lately in Linder and Armero (2007, 2009) where the new finite elements that incorporate strong discontinuities with linear interpolations of the displacement jumps are developed in infinitesimal deformation and finite deformation range and in multiphysics problems in Linder et al. (2011a); Linder and Miehe (2012); Linder (2012). These strong discontinuities can have an arbitrary orientation and also propagate through the finite element, which makes it much easier to capture a propagating crack or shear band without resorting to re-meshing methods or dependence on the underlying mesh. This class of method is generally referred to as the embedded finite element method or simply EFEM.

The major advantage of the embedded finite element method lies in the local nature of additional degrees of freedom which are introduced to represent the strong discontinuity through a finite element. These local (additional) degrees of freedom are statically condensed out at the element level preserving the global system of equations i.e. the total degrees of freedom remain the same as in an element without the discontinuity. The numerical implementation of such a methodology can be easily done by small modifications to any existing finite element code.

The local nature of the newly introduced additional degrees of freedom of a strong discontinuity is a direct outcome of a key feature of the embedded finite element method by which a strong discontinuity is incorporated locally in the neighborhood of a material point where the localization in the form of the loss of ellipticity condition has been detected. The given boundary value problem and its solution then represents the global problem where incremental nodal displacements (global degrees of freedom) are obtained by a standard iterative procedure. The introduction of a strong discontinuity locally represents the local problem where the dissipation mechanisms are captured by a cohesive damage law representing a constitutive relation between the displacement jumps and the tractions along the strong discontinuity. An equilibrium between the tractions coming from the bulk of the solid and tractions resulting from the damage law along the strong discontinuity surface is enforced to obtain the incremental local degrees of freedom (displacement jumps).

### 1.2 Goals

The behavior of solids at failure critically depends on the nature of the external loading. For a higher rate of dynamic loading, the strain energy release rate (dissipation) associated with a single propagating crack may be insufficient leading to the formation of multiple cracks in the solids. Numerical modeling of such a failure phenomena, called as crack branching in brittle materials under dynamic loading, requires the resolution of multiple strong discontinuities through a single finite element to correctly predict the behavior of the failing solid.
To achieve this, a numerical method, called as the multilevel embedded finite element method is proposed in this work, which treats the finite element where the crack branching initiation criterion has been met in the incremental time step \([t_n, t_{n+1}]\) as a new boundary value problem referred to as the sub-boundary value problem. The boundary of this finite element is subjected to certain kinematic constraints in the sub-boundary value problem and is subsequently discretized during the runtime into a number of finite elements, referred to as sub-elements. The most important consideration in the proposed methodology is that the multiple strong discontinuities in the finite element are incorporated by introducing the jumps in the displacement fields of the sub-elements in the sub-boundary value problem. The state of branching i.e. the angle of branching and the number of branching cracks is then the natural outcome of the solution of the sub-boundary value problem.

This is done by partitioning of the nodes of the discretized finite element into the nodes lying on the boundary and the nodes lying in the interior of the finite element where the nodes on the boundary are subjected to a deformation constraint. Finally, the virtual energies due to the admissible variations at different levels of discretization are matched to derive the internal force vector and the stiffness matrix associated with the finite element in terms of the internal forces and the stiffness matrices of the sub-elements associated with the boundary.

### 1.3 Outline

The rest of the work is divided into four chapters. In Chapter 2, the embedded finite element method is presented with a brief introduction to the general framework of finite element method in Sect. 2.2. The incorporation of strong discontinuities in the continuum and finite element setting are shown in Sect. 2.3.1 and Sect. 2.3.2 respectively. In Chapter 3, the physical phenomena of crack branching in brittle materials is discussed with the challenges involved in its numerical modeling briefly outlined in Sect. 3.3. In Chapter 4, a new methodology called as ‘the multilevel embedded finite element method’ is introduced which is the main focus of this work. In Chapter 5, the experiments of Sharon et al. (1996) are simulated using the new method (Chapter 4) and compared with numerical results in Raina and Linder (2010).
2 The Embedded Finite Element Method

In this chapter, the embedded finite element method, also referred to as EFEM, as a computational tool to model the failure of solids discretely by virtue of local dissipation along the strong discontinuities will be introduced. The discussion follows the recent work by Linder and Armero (2007, 2009).

2.1 Introduction

The simulation of solids at failure in the form of cracks or shear bands pose a challenge in the numerical modeling due to the non-smooth solution of the underlying problem. As mentioned earlier, the key approach lies in capturing the strain localization of the failing solid which is achieved in the embedded finite element method by introducing a jump $[u]$ in the displacement field $u$, a strong discontinuity, of the failing solid. The strong discontinuity is introduced locally in the neighborhood of a material point where the failure has taken place, thereby, defining a total displacement field in the region in terms of a continuous part associated with the global displacement field and a discontinuous part associated with the strong discontinuity.

The problem of the singular strain field, arising due to the gradient of the local displacement jump, is resolved by the characterization of proper kinematics associated with the finite element where a strong discontinuity has developed i.e. the identification of correct separation modes associated with the finite element as presented in Sect. 2.3.3. A piece-wise linear interpolation of displacement jumps, in both the normal and tangential directions, was considered in Linder and Armero (2007) for the quadrilateral finite elements under quasi-static loading conditions. This lead to the development of an enhanced strain operator and a locking-free property, which in Linder and Armero (2009) was extended to the dynamic setting to tackle problems associated with dynamic fracture. The resulting stresses in the bulk are now defined in terms of the total strain field which is the sum of continuous global strain field and an enhanced strain field associated with the strong discontinuity.

First, a standard initial boundary value problem will be formulated in the continuum setting in Sect. 2.2.1 and in the discrete setting in Sect. 2.2.2. Later, the incorporation of the strong discontinuity will be presented in the continuum framework in Sect. 2.3.1 and then in the finite element framework in Sect. 2.3.2. Finally, the numerical implementation of the embedded finite element method allowing for a static condensation of the enhanced local variables will be shown in Sect. 2.4.

2.2 The Initial Boundary Value Problem

A standard initial boundary value problem is first summarized within the continuum framework discussed in Sect. 2.2.1. Its discrete approximation is outlined later in Sect. 2.2.2.
2.2.1 The Continuum Framework

A solid body $B$, represented by a fixed open domain $B \subset \mathbb{R}^{n_{\text{dim}}}$ for $n_{\text{dim}} = 1..3$, under a state of equilibrium is considered here. In the infinitesimal deformation setting, the equilibrium position of $B$ is characterized by a displacement field $u : B \times [0, T] \rightarrow \mathbb{R}^{n_{\text{dim}}}$ in time $t \in [0, T]$. Let $x$ represent the position of a material point in the body $B$ with the density at each material point denoted by $\rho(x, t)$. The associated velocity field $v(x, t) = \dot{u}(x, t)$ and acceleration field $a(x, t) = \ddot{u}(x, t)$ are computed for inertia effects where $(\dot{\cdot})$ represents the spatial time derivative $\frac{\partial}{\partial t}$ of the spatial field. We consider the associated infinitesimal symmetric strain tensor $\varepsilon : B \times [0, T] \rightarrow \mathbb{R}^{n_{\text{dim}} \times n_{\text{dim}}}$ for the standard gradient operator $\nabla$ in $x$ and the corresponding symmetric stress tensor $\sigma : B \times [0, T] \rightarrow \mathbb{R}^{n_{\text{dim}} \times n_{\text{dim}}}$, where symmetry results in the a priori fulfillment of the balance of angular momentum, related to strain tensor $\varepsilon$ by a constitutive relation describing the material response.

Consider the body $B$ subjected to a volumetric body force $b : B \rightarrow \mathbb{R}^{n_{\text{dim}}}$ per unit mass and displacements $u_0 : \partial_u B \rightarrow \mathbb{R}^{n_{\text{dim}}}$ on the displacement boundary $\partial_u B \subset \partial B$ representing the Dirichlet boundary conditions and tractions $t_0 : \partial_t B \rightarrow \mathbb{R}^{n_{\text{dim}}}$ on the traction boundary $\partial_t B \subset \partial B$ representing the Neumann boundary conditions (see Fig. 1). The body $B$ is also subjected to an imposed velocity $v_0 : \partial_u B \times \{t = 0\} \rightarrow \mathbb{R}^{n_{\text{dim}}}$ on the displacement boundary $\partial_u B$ representing the initial conditions. The split of the domain boundary $\partial B$ into displacement boundary $\partial_u B$ and traction boundary $\partial_t B$ is such that $\overline{\partial_u B \cup \partial_t B} = \partial B$ and $\partial_u B \cap \partial_t B = \emptyset$.

The strong form of the problem is to find the displacement field $u : B \times [0, T]$ satisfying

\[
\begin{align*}
\text{div } \sigma + \rho b &= \rho \dot{u} \quad \text{in} \quad B \\
\sigma &= C \varepsilon \\
u &= u_0 \quad \text{on} \quad \partial_u B \\
\dot{u} &= v_0 \quad \text{on} \quad \partial_u B \\
\sigma n &= t_0 \quad \text{on} \quad \partial_t B
\end{align*}
\]

(1)

where the first equation in (1) represents the balance of linear momentum (for dynamic problem), the second equation in (1) represents the constitutive equation defining the material response to applied loads and the third, fourth and fifth equations in (1) represent the boundary and initial conditions for the problem to be well-posed. Obtaining the exact solution of the strong form of the problem can be virtually impossible for complex domains. However an approximate solution can be achieved by defining the weak form of the above problem in (1).

For defining the weak form, we start with the characterization of two classes of functions. We first introduce the trial solution $\mathbf{u}$ which we require to satisfy the Dirichlet boundary conditions and initial conditions. The trial functions are introduced such that their derivatives are square-integrable. Therefore for $\mathbf{u}$ to be a trial function

\[
\int_B (\nabla \mathbf{u})^2 \, dV < \infty \quad \text{i.e.} \quad \mathbf{u} \in H^1
\]

(2)

where $H^1$ is a class of Hilbert space and functions belonging to this space are called
$H^1$-functions. We denote the set of such trial solutions as $S$ where

$$S = \{ u \mid u \in H^1, \, u(\partial_uB) = u_0, \, \dot{u}(\partial_uB) = v_0 \}. \quad (3)$$

We then introduce the second class of functions called the weighting functions or simply variations $\delta u$ which are similar to the trial functions except now these variations have to satisfy the homogeneous Dirichlet boundary conditions, i.e. the variations vanish on the Dirichlet boundary. We denote the set of such variations as $V$ where

$$V = \{ \delta u \mid \delta u \in H^1, \, \delta u(\partial_uB) = 0 \}. \quad (4)$$

This choice of trial functions and variations satisfy a priori the third and fourth equation of the strong form (1). Also the constitutive equation in (1) will be strongly satisfied. Therefore introducing the approximations $u \in S$ and $\delta u \in V$ into the first and fifth equation of strong form (1) to obtain a weak form results in residuals as

$$\text{div} \, \sigma + \rho b - \rho \ddot{u} = 0 \quad \text{in} \quad B \quad \quad (5)$$
$$\sigma n - t_0 = 0 \quad \text{on} \quad \partial_lB \quad \quad (6)$$

Scalar multiplication of (5) and (6) by the variations and integration over their respective domains yields

$$\int_B (\text{div} \, \sigma + \rho b - \rho \ddot{u}) \cdot \delta u \, dV = 0 \quad \text{and} \quad \int_{\partial_lB} (\sigma n - t_0) \cdot \delta u \, dA = 0 \quad (7)$$

Application of Green’s theorem to the first part in (7) results in

$$\int_B \text{div} \, \sigma \cdot \delta u \, dV + \int_B (\rho b - \rho \ddot{u}) \cdot \delta u \, dV = 0$$
$$\int_B \text{div}(\sigma \delta u) \, dV - \int_B \sigma : \nabla(\delta u) \, dV + \int_B (\rho b - \rho \ddot{u}) \cdot \delta u \, dV = 0$$
$$\int_{\partial_lB} \sigma n \cdot \delta u \, dA - \int_B \sigma : \nabla(\delta u) \, dV + \int_B (\rho b - \rho \ddot{u}) \cdot \delta u \, dV = 0 \quad (8)$$

where the use of the property $\delta u = 0$ on $\partial_uB$ and the divergence theorem is made in the first part of equation (8). Now from the second part of (7) we obtain

$$\int_{\partial_lB} \sigma n \cdot \delta u \, dA = \int_{\partial_lB} t_0 \cdot \delta u \, dA \quad (9)$$

Substitution of (9) into (8) and with the rearrangement of terms yields

$$\int_B \rho \ddot{u} \cdot \delta u \, dV + \int_B \sigma : \nabla(\delta u) \, dV = \int_B \rho b \cdot \delta u \, dV + \int_{\partial_lB} t_0 \cdot \delta u \, dA \quad (10)$$

which represents the weak statement of the problem discussed in (1). Formulations of this type are often referred to as principle of virtual work. Based upon the weak formulation in (10) an approximate solution of the mechanical initial boundary value problem can be obtained e.g. by finite element method as discussed in the next section.
2.2.2 The Finite Element Framework

We discussed the continuum problem in Sect. 2.2.1 corresponding to the standard initial boundary value problem of the domain $B$. The weak form (10) was obtained from the strong form (1) which now facilitates the discretization of the domain $B$ into finite elements $B_h$ as shown in this section. We first construct the finite-dimensional approximations of $S$ and $V$ and denote them as $S^h$ and $V^h$ respectively, where

\[ S^h \subset S \quad \text{(i.e., if } u^h \in S^h, \text{ then } u^h \in S) \]  
\[ V^h \subset V \quad \text{(i.e., if } \delta u^h \in V^h, \text{ then } \delta u^h \in V) \]  

The finite element approximation of the displacement field is then written as

\[ u(x, t) \approx u^h(x^h, t) = \sum_{A=1}^{n_{\text{node}}} N_A(x^h) d_A(t) = N d \]  

resulting in the approximation of its variation as

\[ \delta u \approx N \delta d \]  

where the use of standard isoparametric concept for the deformation and the geometry is made. Here $N^A$ is a set of shape functions prescribed in terms of approximated geometry $x^h(= N^A x_A)$ where $x_A$ are discrete (nodal) positions in $B^h$ and $d_A(t)$ is the set of discrete (nodal) displacements associated with the nodes $A = 1..n_{\text{node}}$ of the discretized domain $B^h = \bigcup_{e=1}^{n_{\text{elem}}} B^h_e$. The velocity field $v^h(x^h, t)$ and acceleration field $a^h(x^h, t)$ are approximated in terms of nodal velocity $v_A(t)$ and nodal acceleration $a_A(t)$ in the similar fashion as the displacement field shown above. From here on the approximated global displacement, velocity and acceleration arrays will be represented by $u$, $v$ and $a$ respectively.

The strain field is approximated by using a generic B-bar matrix which encompasses the possibility of using different finite element formulations (displacement, mixed, assumed, enhanced) and accordingly written as

\[ \varepsilon(u(x, t)) \approx \varepsilon^h(d) = \sum_{A=1}^{n_{\text{node}}} B^A(x^h) d_A(t) = B d \]  

resulting into the definition of

\[ \nabla(\delta u) \approx B \delta d \]  

for the nodal displacement variations $\delta d$. By introducing the discrete counterpart of displacement field and its variation into the weak form (10) for a generic finite element
\[ B^h \subset B^b, \text{ we get} \]
\[ \int_{B^b} \rho \ddot{u} \cdot \delta u^h \, dV + \int_{B^b} \sigma : \nabla (\delta u^h) \, dV = \int_{\partial B^b} \rho \dot{u} \cdot \delta u^h \, dA \]
\[ \int_{B^b} (N \delta \dot{d})^T \cdot \rho (N \ddot{d}) \, dV + \int_{B^b} (B \delta \dot{d})^T : \sigma \, dV = \int_{\partial B^b} (N \delta \dot{d})^T \cdot \rho \dot{b} \, dV + \int_{\partial B^b} (N \delta \dot{d})^T \cdot \dot{t}_0 \, dA \]
\[ \delta \ddot{d}^T \cdot \left[ \int_{B^b} \mathbf{M} \mathbf{a}^e + \int_{B^b} \mathbf{B}^T \mathbf{f}_{\text{int}} \right] = \delta \dot{d}^T \cdot \left[ \int_{B^b} \mathbf{N} \rho \mathbf{b} \, dV + \int_{B^b} \mathbf{N} t_0 \, dA \right] \] (17)

where (17) has to hold for any arbitrary variation \( \delta d \neq 0 \) and \( \mathbf{f}_{\text{int}}, \mathbf{f}_{\text{ext}}, \mathbf{M}^e \) and \( \mathbf{a}^e \) represent the internal force vectors, external force vectors, mass matrix and the nodal acceleration vector associated with the finite element \( B^h \). The same holds true for the discretized domain \( B^b \) where by assembling the element contributions we finally arrive at global discrete set of equations written in the form of residual as

\[ \mathbf{R}(\mathbf{u}, t) = \mathbf{f}_{\text{ext}}(t) - \mathbf{f}_{\text{int}}(\mathbf{u}, t) - \mathbf{M} \mathbf{a} = 0 \] (18)

for the global external and internal force vectors written as

\[ \mathbf{f}_{\text{ext}} = \int_{B^b} \mathbf{N}^T \rho \mathbf{b} \, dV + \int_{\partial B^b} \mathbf{N}^T \dot{t}_0 \, dA \quad \text{and} \quad \mathbf{f}_{\text{int}} = \sum_{e=1}^{n_{\text{elem}}} \left( \int_{B^h} \mathbf{B}^T \mathbf{f} \, dV \right) \] (19)

and the assembled mass matrix \( \mathbf{M} \) given in terms of the element contributions \( \mathbf{M}^e \) as

\[ \mathbf{M} = \sum_{e=1}^{n_{\text{elem}}} \int_{B^h} \rho \mathbf{N}^T \mathbf{N} \, dV. \] (20)

The governing equation (18) is a second order transient equation which is solved by a standard implicit time integration scheme such as Newmark's method as written below for the time step \( t_{n+1} \).

\[ \mathbf{R}(\mathbf{u}_{n+1}, t_{n+1}) = \mathbf{f}_{\text{ext}}(t_{n+1}) - \mathbf{f}_{\text{int}}(\mathbf{u}_{n+1}, t_{n+1}) - \mathbf{M} \mathbf{a}_{n+1} = 0 \] (21)

A linear ansatz is introduced to compute velocity and displacement at time step \( t_{n+1} \) as

\[ \mathbf{v}_{n+1} = \mathbf{v}_n + \Delta t \left\{ (1 - \gamma)\mathbf{a}_n + \gamma \mathbf{a}_{n+1} \right\} \] (22)

and

\[ \mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t \mathbf{v}_n + \Delta t^2 \left\{ (0.5 - \beta)\mathbf{a}_n + \beta \mathbf{a}_{n+1} \right\} \] (23)

where Newmark parameters \( \gamma \) and \( \beta \) are independent of each other controlling the stability and numerical dissipation of the governing equation (21). Rearranging (23) yields the relation for acceleration at \( t_{n+1} \) as

\[ \mathbf{a}_{n+1} = \frac{1}{\beta \Delta t^2} (\mathbf{u}_{n+1} - \mathbf{u}_n) - \frac{1}{\beta \Delta t} \mathbf{v}_n - \left( \frac{1}{2 \beta} - 1 \right) \mathbf{a}_n \] (24)

which is substituted back into (21) to obtain the displacement at \( t_{n+1} \) by an iterative procedure. The standard value of \( \beta = 0.25 \) and \( \gamma = 0.5 \) is used in the numerical simulations to be presented later in Chapter 5.
The Embedded Finite Element Method

Figure 1: The global problem is illustrated in the continuum setting by the initial boundary value problem of body $B$ with applied loadings and boundary conditions. The local problem is illustrated by the small neighborhood $B_x \subset B$ of $x$ where a strong discontinuity $\Gamma_x$ with unit normal $n$ in the form of a jump $[u_n]$ in the displacement field is detected.

2.3 The Incorporation of Strong Discontinuities

Section 2.2 described the initial boundary value problem of a body $B$ with the assumption that no localization in the form of cracks or shear bands takes place at any material point $x \in B$ i.e. the displacement field $u$ in equations (10) and (18), representing the continuum and discretized frameworks respectively, is smoothly continuous. In this section our goal is to present the modeling of the localized failure in solids by the incorporation of jumps in the displacement field, called as strong discontinuities, both in the continuum (Sect. 2.3.1) as well as discrete (Sect. 2.3.2) setting. A pivotal aspect in the realization of this goal is the resolution of the kinematics of these discontinuous solutions by local considerations associated to the local problem, maintaining the structure of the global problem.

This argument of separation of the given problem at hand into a global and a local problem was originally presented in Armero (1999, 2001) and briefly discussed in Sect. 2.3.1 leads to an efficient resolution of strong discontinuities through local enhancements of the finite element in the discrete setting as shown in Sect. 2.3.2.

2.3.1 The Continuum Modeling of Strong Discontinuities

We first identify the standard initial boundary value problem and its solution as presented in Sect. 2.2 as the the global problem and the associated displacement field $u(x, t)$ as the global displacement field.

Let a strong discontinuity (failure in the form of a crack or a shear-band) $\Gamma_x \subset \mathbb{R}^{n_{\text{dim}}-1}$ with a unit normal $n$ be now detected by some failure criterion at a material point $x \in B$. We consider a small neighborhood $B_x \subset B$ of $x$ where the strong discontinuity is incorporated in the form of a jump in the displacement field $[u_n] : \Gamma_x \rightarrow \mathbb{R}^{n_{\text{dim}}}$ which now represents the local problem at hand and the associated discontinuous displacement field $\tilde{u}(\lbrack u_n \rbrack)$ as the local displacement. To this end, the kinematics of deformation in the region $B_x$ is now defined in terms of a total displacement field

$$u_n(x, t) = u(x, t) + \tilde{u}(\lbrack u_n \rbrack, t) \quad \text{in} \quad B_x \quad (25)$$

as the sum of the displacements coming from global problem $u(x, t)$ and local problem
\( \tilde{u}(\lbrack u \rbrack, t) \). See Fig. 1 for an illustration.

The *total strain* field associated with the total displacement \( u_\mu \) is now written as

\[
\varepsilon_\mu = \varepsilon(u) + \tilde{\varepsilon}(\tilde{u}) \quad \text{in} \quad B_x \setminus \Gamma_x
\]

which is defined in the bulk of the neighborhood \( B_x \setminus \Gamma_x \) in terms of the global strain field \( \varepsilon(u) \) and the local strain field \( \tilde{\varepsilon}(\tilde{u}) \) arising from the displacement jump \( \lbrack u \rbrack \) defining a singular distribution on the surface \( \Gamma_x \) (i.e. a Dirac delta function).

An important attribute here is that the total strain field (26) characterizes the bulk response of the material and accordingly defines the stress field \( \sigma \) in the bulk depending upon the constitutive material model used to define the material behavior. The numerical simulations presented in Chapter 5 simply follow the linear elastic isotropic response in the bulk where there is no failure taking place, for the brittle material polymethylmethacrylate (PMMA), which is written as

\[
\sigma = C \varepsilon_\mu \quad \text{in} \quad B_x \setminus \Gamma_x \quad \text{for} \quad C = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}
\]

in terms of the usual Lamé constants \( \lambda \) and \( \mu \) and the second- and fourth- order identity tensors \( \mathbf{1} \) and \( \mathbf{I} \) respectively.

In addition to the characterization of the material response in the bulk of the material, a constitutive relation relating the driving tractions \( t_\Gamma \) and the displacement jumps \( \lbrack u \rbrack \) along the strong discontinuity surface \( \Gamma_x \) is employed to fully characterize the material behavior. This is accomplished by incorporating a cohesive law which accounts for the dissipation of strain energy during localization (due to damage, plastic, poroplastic etc. effects) based on the fracture energy \( G_f \) of the material under consideration.

The introduction of a new local field \( \lbrack u \rbrack \) in (25) requires an additional equation for the closure. This is achieved by enforcing an equilibrium along the strong discontinuity surface \( \Gamma_x \) between the driving tractions \( t_\Gamma \) resulting from the applied cohesive law and the tractions resulting from the stresses in the bulk of the material. This local equilibrium can be written in the weak form as

\[
\int_{\Gamma_x} \delta\lbrack u \rbrack \cdot (\sigma n - t_\Gamma) \, dA = 0 \quad (28)
\]

for all admissible variations \( \delta\lbrack u \rbrack : \Gamma_x \to \mathbb{R}^{n_{\text{dim}}} \) of the jumps \( \lbrack u \rbrack \) in terms of trace of stresses on the discontinuity surface \( \Gamma_x \). From the physical considerations, it is to be observed that no mass is associated to the discontinuity surface \( \Gamma_x \). Another important consideration is the local nature of the equation (28) holding in \( B_x \) (\( \Gamma_x \subset B_x \)) and not the global domain \( B \) which explicitly allows for the solution of the local field \( \lbrack u \rbrack \) in terms of the global displacement field \( u \) in the limit

\[
h_x = \frac{\int_{B_x} dV}{\int_{\Gamma_x} dA} \to 0 \quad (29)
\]

i.e. of the vanishing scale of local problem \( B_x \), which is the case of interest in the numerical setting of the problem.
2.3.2 The Finite Element Modeling of Strong Discontinuities

Section 2.3.1 described the continuum framework of the incorporation of a strong discontinuity through a local neighborhood of a material point. This notion can be extended to the finite element framework where the local neighborhood $B_x \subset B$ is identified by a finite element $B_h^e \subset B^h$. The idea is consistent with the limit (29) of vanishing local problem scale as can be easily understood by limiting process of the refined finite element meshes. In addition to this, the local nature is fully exhibited by the fact that the strong discontinuity lies in the finite element domain leading to an efficient local structure of the final method.

We consider a single finite element $B_h^e$ where a strong discontinuity $\Gamma_h^e$ has been detected and activated. We define a unit normal $n$ and a unit tangent $m$ at the mid-point $x_\Gamma$ to the discontinuity surface $\Gamma_h^e$ and a local coordinate $s$ along $\Gamma_h^e$ from the mid-point $x_\Gamma$. In this context we consider only straight segments of discontinuity crossing the element fully such that $\Gamma_h^e \in B_h^e$. See Fig. 2 for an illustration.

The finite element approximation of the global fields of displacement and strain as introduced in equations (25) and (26) will be done in the standard manner as shown in equations (13) and (15) respectively. Now the discontinuity jumps need to be approximated in the finite element setting which are locally interpolated along $\Gamma_h^e$ as

$$[u_\mu](s) \approx [u_\mu^h](s) = J(s)\xi$$

by introducing a jump interpolation matrix $J(s)$ applied on the local element parameters $\xi$. It is to be emphasized here that the local parameters $\xi$ are defined independently at the element level through which a discontinuity has propagated showing the local nature of the underlying methodology. Thus the approximated counterpart of total displacement field (25) within a finite element can be written as

$$u_\mu^h = Nd + J\xi \quad \text{in} \quad B_h^e.$$  \hspace{1cm} (31)

Similarly the finite element approximation of the total strain field (26) takes the form

$$\varepsilon_\mu^h = \bar{B}d + G(c)\xi \quad \text{in} \quad B_h^e \setminus \Gamma_h^e$$

in terms of a newly introduced linear enhanced strain operator $G(c)$ called as ‘compatibility operator’ acting on the local element parameters $\xi$ and defines the effect of strain field due to displacement jump in the bulk of the finite element $B_h^e \setminus \Gamma_h^e$. The strains in (32) now define the stresses $\sigma$ appearing in the global set of discrete equation (18) for the element $B_h^e$ through a constitutive relation e.g. equation (27).

The local equilibrium equation (28) identifying the driving tractions $t_\Gamma$ along the discontinuity with the tractions arising from the bulk stresses $\sigma$ is approximated as

$$r_{enh}^e = -\int_{B_h^e} G^T(c)\sigma \, dV - \int_{\Gamma_h^e} J^T t_\Gamma \, dA = 0, \quad B_h^e \in \mathcal{E}_{disc}$$

in terms of an ‘equilibrium operator’ $G(c)$ defined in the bulk of the element $B_h^e$ after introducing the discrete variations of the jump (30) in terms of the jump interpolation matrix $J$. The minus sign in (33) is just for the convenience in the numerical implementation.
Figure 2: Discrete global problem is illustrated by the discretized body $B^h$ with applied loadings and boundary conditions on the nodes. The finite element $B^h_e \subset B^h$ with a strong discontinuity $\Gamma^h_e$ through it in the form of discrete displacement jump $\lbrack u^h \rbrack$ now represents the discrete local problem.

From the definition of the local residual $r^e_{\text{enh}}$ in (33), $G_{(e)}$ can be viewed as a projection operator projecting the stresses $\sigma$ from the bulk of the element $B^h_e$ to the associated tractions along the discontinuity $\Gamma^h_e$. See Linder and Armero (2007) for more details on the definition of $G_{(e)}$. Furthermore, it is to be noticed that (33) is defined locally only for the set of elements $E_{\text{disc}}$ where a discontinuity has been activated. The definitions of the jump interpolation matrix $J$ and the compatibility operator $G_{(c)}$ introduced in (31) and (32) respectively should complete the finite element formulation of the incorporation of strong discontinuities and is briefly discussed in the design of new finite elements in Sect. 2.3.3.

2.3.3 Finite Element Design

In Linder and Armero (2007), a piece-wise linear interpolation of the discontinuity jumps has been established and subsequently the jump interpolation matrix $J$ and enhanced strain operator $G_{(c)}$ devised resulting in an efficient and locking free property of the new finite element. One can observe that $G_{(c)}$ must capture the kinematics of the finite element with an embedded discontinuity to correctly resolve the singular strains associated with the displacement jumps. This is obtained by the incorporation of certain separation modes directly into the finite element such that the given mode derives the exact solution of the resulting total strain field with a locking-free representation of the failing solid.

The two basic modes which are developed and used in Linder and Armero (2007) are the constant separation mode and the linear separation mode. See Fig. 3 for an illustration. For instance, the linear separation mode consists of one side of the element across the discontinuity segment $\Gamma^h_e$ separating and infinitesimally rotating while uniformly stretching in the tangential direction to the discontinuity. No other strains must appear, hence avoiding the appearance of spurious stresses transferring across the discontinuity. Therefore the jumps in the linear separation mode can be written as

$$\lbrack u^h \rbrack (s) = \xi^{(0m)}n + \xi^{(0m)}m + s\xi^{(1n)}n + s\xi^{(1m)}m$$

(34)

in terms of four local element parameters (local degrees of freedom) $\xi$ resulting in the
The Embedded Finite Element Method

\[ \mathbf{B} - \mathbf{B}_h + \mathbf{\Gamma} \mathbf{x} \mathbf{m} \]

\[ \mathbf{u}_h \mathbf{\mu} \]

\[ \langle 0 \rangle \xi \langle 0n \rangle \xi \langle 0m \rangle \]

\[ \langle 0 \rangle_1 \]

\[ \langle 0 \rangle_2 \]

\[ \langle 0 \rangle_3 \]

\[ \langle 0 \rangle_4 \]

Figure 3: Illustration of different separation modes incorporated here within a four-node quadrilateral finite element \( \mathcal{B}_e^h \) used for designing the enhanced strain operator \( \mathbf{G}_{(c)} \).

definition of the jump interpolation matrix \( \mathbf{J} \) as

\[ \mathbf{J}(s) = [\mathbf{n} \ \mathbf{m} \ \mathbf{s} \mathbf{n} \ \mathbf{s} \mathbf{m}] \] (35)

To derive the enhanced strain operator for the linear separation mode, the resulting total strain in \( \mathcal{B}_e^h \) for the normal and tangential separations is taken into account yielding the definition of \( \mathbf{G}_{(c)} \) as

\[ \mathbf{G}_{(c)}^{(1)} = [\mathbf{G}_{(c)}^{(0n)} : \mathbf{G}_{(c)}^{(0m)} : \mathbf{G}_{(c)}^{(1n)} : \mathbf{G}_{(c)}^{(1m)}] \] (36)

where superscript \((1)\) denotes the linear enhanced operator and the sub-matrices with superscripts \((0n), (0m), (1n)\) and \((1m)\) represent the individual enhanced operators for constant normal separation, constant tangential separation, linear normal separation and linear tangential separation respectively with

\[ \mathbf{G}_{(c)}^{(0n)} = - \sum_{A \in \mathcal{B}_e^h} \mathbf{B}^A \mathbf{n} \]

\[ \mathbf{G}_{(c)}^{(0m)} = - \sum_{A \in \mathcal{B}_e^h} \mathbf{B}^A \mathbf{m} \]

\[ \mathbf{G}_{(c)}^{(1n)} = - \sum_{A \in \mathcal{B}_e^h} \mathbf{B}^A (\mathbf{n} \otimes \mathbf{m}) \mathbf{\bar{x}}_A \]

\[ \mathbf{G}_{(c)}^{(1m)} = (\mathbf{m} \otimes \mathbf{m}) \mathcal{H}_{\Gamma} - \sum_{A \in \mathcal{B}_e^h} \mathbf{B}^A (\mathbf{m} \otimes \mathbf{m}) \mathbf{\bar{x}}_A \] (37)

written in terms of summations of the B-Bar matrix of the nodes on one side of the discontinuity multiplied with corresponding combinations of the unit normal- and tangential vectors \( \mathbf{n} \) and \( \mathbf{m} \). Use of the standard Heaviside function \( \mathcal{H}_{\Gamma} \) and the abbreviation \((\mathbf{n} \otimes \mathbf{m})^A = \mathbf{n} \otimes \mathbf{m} - \mathbf{m} \otimes \mathbf{n} \) is made as well as of \( \mathbf{\bar{x}}_A = \mathbf{x}_A - \mathbf{x}_\Gamma \) with \( \mathbf{x}_A \) representing the coordinates of node \( A \) and \( \mathbf{x}_\Gamma \) as the center of the discontinuity. A detailed derivation of the quantities in (37) can be found in Linder and Armero (2007). Such a construction of the enhanced strain operator \( \mathbf{G}_{(c)} \) allows the final numerical formulation to be independent of the local discontinuous displacement field in element \( \mathcal{B}_e^h \in \mathcal{E}_{\text{disc}} \) as shown in equation (25).
2.4 Numerical Implementation

The finite element formulation with embedded discontinuity discussed in the previous sections laid an emphasis on preserving the structure of a standard finite element code and at the same time made the formulation computationally efficient. This leads to a simplistic implementation of the finite element code as discussed in this section. In particular, it involves the solution of the governing equations (21) and (33) representing the discrete global and local residual equations respectively.

2.4.1 Linearization of Discrete Governing Equations

The governing equations (21) and (33) represent the spatially discretized non-linear transient equations of second order for the dynamic case of interest here. The time discretization of these equations is done by a standard time integration scheme as discussed briefly in Sect. 2.2.2 (Eq. (22–24)). Here a standard Newton-Raphson scheme is considered for the solution of the non-linear algebraic system of equations written here at time step $t_{n+1}$ as

$$
R(d_{n+1}, \xi_{n+1}) = f_{ext}(t_{n+1}) - \sum_{\epsilon=1}^{n_{elem}} \left( \int_{B^h_e} B^T \sigma_{n+1} d V \right) - M a_{n+1} \quad (38)
$$

$$
r_{enh}^e(d_{n+1}^e, \xi_{n+1}^e) = - \int_{B^h_e} G^T_{(e)} \sigma_{n+1} d V - \int_{r_{enh}^e} J^T t_{\Gamma_{n+1}} d A, \quad B^h_e \in \delta_{disc} \quad (39)
$$

where their linearized versions are iteratively brought to zero to obtain incremental updates of the dependent variables. See Fig. 4 for an illustration. The linearization of the set of equations in (38) and (39) is performed about the last known equilibrium position $\{d_{n+1}^k, \xi_{n+1}^k\}$ and with respect to the dependent variables $d^k$ and $\xi^k$ where superscript $k$ indicates the iteration index. Further equations are assumed to be written at time $t_{n+1}$ and henceforth dropped while the iteration index $k$ is retained. From (38), we write the linearized form as

$$
\text{Lin}_{d^k, \xi^k} R(d^{k+1}, \xi^{k+1}) = R(d^k, \xi^k) + \frac{\partial R}{\partial d} \bigg|_{d^k} (d^{k+1} - d^k) + \frac{\partial R}{\partial \xi} \bigg|_{\xi^k} (\xi^{k+1} - \xi^k) = 0
$$
Using the relation (32) in $\sigma = \mathbb{C}\varepsilon^h = \mathbb{C}(\tilde{\mathbf{B}}d + \mathbf{G}(\xi)\xi)$ and the relation (24) in acceleration field $\mathbf{a}$ and the fact that the external force vector $\mathbf{f}_{\text{ext}}$ (19) is independent of the variables $d$ and $\xi$ yields

$$
\frac{\partial R}{\partial \mathbf{d}} \bigg|_{d^k} = -\frac{\partial}{\partial d^k} \left\{ \sum_{e=1}^{n_{\text{elem}}} \mathbf{A} \left( \int_{B_e^k} \tilde{\mathbf{B}}^T \mathbb{C}(\tilde{\mathbf{B}}d^k + \mathbf{G}(\xi)\xi) dV \right) + \mathbf{M} \right\}
$$

$$
= -\sum_{e=1}^{n_{\text{elem}}} \left( \int_{B_e^k} \tilde{\mathbf{B}}^T \mathbb{C} \mathbf{d} dV \right) + \frac{1}{\beta \Delta t^2} \mathbf{M}
$$

$$
= -\sum_{e=1}^{n_{\text{elem}}} \left( K_{\xi d}^k + \frac{1}{\beta \Delta t^2} \mathbf{M} \right) = -\sum_{e=1}^{n_{\text{elem}}} \mathbf{K}_{\xi d}^k
$$

and in a similar manner, one can also write

$$
\frac{\partial R}{\partial \xi} \bigg|_{\xi^k} = -\frac{\partial}{\partial \xi^k} \left\{ \sum_{e=1}^{n_{\text{elem}}} \mathbf{A} \left( \int_{B_e^k} \tilde{\mathbf{B}}^T \mathbb{C}(\tilde{\mathbf{B}}d^k + \mathbf{G}(\xi)\xi) dV \right) + \mathbf{M} \right\}
$$

$$
= -\sum_{e=1}^{n_{\text{elem}}} \left( \int_{B_e^k} \tilde{\mathbf{B}}^T \mathbb{C} \xi dV \right) = -\sum_{e=1}^{n_{\text{elem}}} \mathbf{K}_{\xi d}^k
$$

The linearization of discrete local residual (39) reads as

$$
\text{Lin}_{d^k, \xi^k} \left( r_{\text{enh}}^e(d^{k+1}, \xi^{k+1}) = r_{\text{enh}}^e(d^k, \xi^k) + \frac{\partial r_{\text{enh}}^e}{\partial d} \bigg|_{d^k} (d^{k+1} - d^k) + \frac{\partial r_{\text{enh}}^e}{\partial \xi} \bigg|_{\xi^k} (\xi^{k+1} - \xi^k) = 0 \right)
$$

Making use of the cohesive law for the relation of tractions $\mathbf{t}_\Gamma$ along the discontinuity with local parameters $\xi$ through (30) yields

$$
\frac{\partial r_{\text{enh}}^e}{\partial d} \bigg|_{d^k} = -\frac{\partial}{\partial d^k} \left\{ \int_{B_e^k} G^T_{(e)} \mathbb{C}(\tilde{\mathbf{B}}d^k + \mathbf{G}(\xi^k)\xi^k) dV - \int_{\Gamma_e^k} J^T(\tilde{\mathbf{C}}_\Gamma \mathbf{J} \xi) dA \right\}
$$

$$
= -\int_{B_e^k} G^T_{(e)} \mathbb{C} \mathbf{d} dV = -\mathbf{K}_{\xi d}^k
$$

and again in the similar manner, we obtain

$$
\frac{\partial r_{\text{enh}}^e}{\partial \xi} \bigg|_{\xi^k} = -\frac{\partial}{\partial \xi^k} \left\{ \int_{B_e^k} G^T_{(e)} \mathbb{C}(\tilde{\mathbf{B}}d^k + \mathbf{G}(\xi^k)\xi^k) dV - \int_{\Gamma_e^k} J^T(\tilde{\mathbf{C}}_\Gamma \mathbf{J} \xi) dA \right\}
$$

$$
= -\left( \int_{B_e^k} G^T_{(e)} \mathbb{C} \xi dV + \int_{\Gamma_e^k} J^T \tilde{\mathbf{C}}_\Gamma \mathbf{J} dA \right) = -\mathbf{K}_{\xi \xi}^k
$$

where the property $\frac{\partial \alpha}{\partial \alpha} = \frac{\partial \alpha}{\partial \alpha} \mathbf{e}_i \otimes \mathbf{e}_j = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{1}$ for $\alpha \in \mathbb{R}^{n_{\text{dim}}}$ is used in equations (40–43) which are substituted back into two linearized boxes to obtain a discrete set of
global and local linear equations as
\[
\mathbf{A} \left[ \begin{array}{c}
\mathbf{K}^{e}_{dd}\Delta d^{k+1} + \mathbf{K}^{e}_{d\xi} \Delta \xi^{k+1}
\end{array} \right] = \mathbf{R}^{e}
\]
for the element incremental arrays \( \Delta d^{k+1} \) and \( \Delta \xi^{k+1} \) with global and local increments given as
\[
d^{k+1} = d^{k} + \Delta d^{k+1} \quad \text{and} \quad \xi^{k+1} = \xi^{k} + \Delta \xi^{k+1}.
\]

2.4.2 Static Condensation

One of the key aspects of the finite element formulation summarized so far is preserving the standard structure of the finite element code by its ability to statically condense the local element parameters on the element level. Thereby requiring the solution of global initial boundary value problem in terms of global displacement field \( \mathbf{d} \). From the second part of linearized equations in (44), local element parameters can be written in terms of the assembled stiffness \( \mathbf{K}^{e}_{dd} \) and the displacement increments \( \Delta d^{k+1} \) as
\[
\Delta \xi^{k+1} = (\mathbf{K}^{e}_{\xi\xi})^{-1} \left[ \mathbf{R}^{e} - \mathbf{K}^{e}_{d\xi} \Delta d^{k+1} \right]
\]
for the elements \( \mathbf{B}^{e}_{h} \in \mathcal{E}_{\text{disc}} \). Inserting (46) into first part of (44) yields
\[
\mathbf{A} \left[ \mathbf{K}^{e}_{dd}\Delta d^{k+1} + \mathbf{K}^{e}_{d\xi} \left( (\mathbf{K}^{e}_{\xi\xi})^{-1} \left[ \mathbf{R}^{e} - \mathbf{K}^{e}_{d\xi} \Delta d^{k+1} \right] \right) \right] = \mathbf{A} \mathbf{R}^{e}
\]
\[
\mathbf{A} \left[ \mathbf{K}^{e}_{dd}\Delta d^{k+1} + \mathbf{K}^{e}_{\xi\xi} \Delta \xi^{k+1} \right] = \mathbf{A} \mathbf{R}^{e}
\]
where the final set of statically condensed equations can now be written as
\[
\mathbf{K}^{e}_{s} \Delta d^{k+1} = \mathbf{R}^{e}
\]
in terms of the assembled stiffness \( \mathbf{K}^{e}_{s} = \mathbf{A} \mathbf{K}^{e}_{\text{eff}} \) and the assembled residual \( \mathbf{R}^{e} = \mathbf{A} \mathbf{R}^{e}_{\text{eff}} \) defined by the effective element contributions as
\[
\mathbf{K}^{e}_{\text{eff}} = \mathbf{K}^{e}_{dd} - \mathbf{K}^{e}_{d\xi} (\mathbf{K}^{e}_{\xi\xi})^{-1} \mathbf{K}^{e}_{\xi d} \quad \text{and} \quad \mathbf{R}^{e}_{\text{eff}} = \mathbf{R}^{e} - \mathbf{K}^{e}_{d\xi} (\mathbf{K}^{e}_{\xi\xi})^{-1} \mathbf{R}^{e}_{\text{enh}}.
\]
2.4.3 Propagation of Strong Discontinuity

The normal \( \mathbf{n} \) to the direction of propagation of a strong discontinuity through a finite element is computed based on the principal direction at each Gauss point \( l \). For a stress tensor \( \mathbf{\sigma} = [\sigma_x \sigma_y \sigma_z \sigma_{xy}]^T \) at each Gauss point \( l \), shown here in Voigt notation for a plane element, the principal stresses are computed as:

\[
\begin{align*}
\eta_1 &= \frac{\sigma_x + \sigma_y}{2}, \\
\eta_2 &= \frac{\sigma_x - \sigma_y}{2}, \\
\sigma_{11} &= \eta_1 + p, \\
\sigma_{22} &= \eta_2 - p
\end{align*}
\]

and the principal angle is computed as

\[
\theta_l = \begin{cases} 
\text{atan2}(\sigma_{xy}, \eta_2) & \eta_2 \neq 0 \\
\pi/4 & \text{else}
\end{cases}
\]

where \( \text{atan2}(a, b) \) is a variation of arctangent \( (\tan^{-1}) \) function. Finally the volume average of principal angles, corresponding to the Gauss points where failure criterion is met, is used as the final value for propagation. However, this may sometimes lead to an incorrect propagation direction due to a certain ambiguity in the code.

To avoid such an instance, two approaches outlined below are implemented in the finite element code. In each approach, a right circular cone with an angular domain of \( 2\theta_A \) is used to restrict the crack propagation. See Fig 5 for an illustration.

**Approach I**- The center-axis for the conical region with the admissible angular domain \( 2\theta_A \) is taken at the default angle \( \theta_d \). The correction is done based on the value of propagation angle computed at the back element \( \theta_b \). e.g.

\[
\begin{align*}
\text{k}k_1 &= \text{abs}(\theta_d - \theta_l) \\
&\text{if (kk1} > \theta_A) \text{ then} \\
&\text{if (\theta_l} < \theta_b) \text{ then} \\
&\quad \theta_l = \theta_b - \theta_A \\
&\text{else} \\
&\quad \theta_l = \theta_b + \theta_A \\
&\text{endif} \\
&\text{endif}
\end{align*}
\]

**Approach II**- Here the center-axis for the conical region with the admissible angular domain \( 2\theta_A \) is taken as the angle computed at back element \( \theta_b \) and the correction is done in the same manner as Approach I. e.g.

\[
\begin{align*}
\text{k}k1 &= \text{abs}(\theta_b - \theta_l) \\
&\text{if (kk1} > \theta_A) \text{ then} \\
&\text{if (\theta_l} < \theta_b) \text{ then} \\
&\quad \theta_l = \theta_b - \theta_A \\
&\text{else} \\
&\quad \theta_l = \theta_b + \theta_A \\
&\text{endif} \\
&\text{endif}
\end{align*}
\]
The quantities introduced in the above two approaches are as follows:

- $\theta_l$ = Angle of crack propagation computed for the current finite element.
- $\theta_d$ = Default angle of crack propagation.
- $\theta_b$ = Angle of crack propagation in the previous finite element.
- $\theta_A$ = Admissible angular domain for the crack propagation.

In the numerical simulations presented in Chapter 5, Approach II is used with the value of $\theta_d = 0^0$ and $\theta_A = 3.5^0$. The choice of $\theta_d$ is motivated by the loading conditions of the boundary value problem and that of $\theta_A$ by the experience of simulation results.
3 Crack Branching in Dynamic Fracture

In this chapter, we will briefly discuss the physical phenomena of crack branching and the various numerical challenges involved in the modeling of such phenomena.

3.1 Introduction

Crack branching is one of the physical aspects of dynamic fracture of engineering materials, particularly brittle materials like inorganic glasses, ceramics etc. The first observations date back to the pioneering experiments of Schardin (1959) with a spurt in experimental observations in the following decades by Kobayashi et al. (1974), Kobayashi and Ramulu (1981), Ravi-Chandar and Knauss (1984a,b), Ramulu and Kobayashi (1985), Fineberg et al. (1991), Sharon et al. (1996), and many more adding to the wealth of experimental observations of dynamic fracture. The general experimental observation is that a single propagating crack will branch into multiple cracks at a certain rate of external loading. Various theoretical frameworks have been established so far to explain this phenomena of crack branching, however, large discrepancies have been found when comparing with experimental results. This can be attributed to the fact that the dynamic fracture is accompanied by certain micro-structural instabilities like formation of micro-crack surfaces which are not taken into account in theoretical framework. The numerical simulation of such a phenomena appears even more challenging due to the dynamic nature of the numerical setting where the inertia effects have to be taken into consideration. Some notable contributions in this field can be attributed to the works of Xu and Needleman (1994), Rafiee et al. (2003), Zhang et al. (2007), Armero and Linder (2009), Linder and Armero (2009) and a few more.

3.2 Mechanism of Crack Branching

Experimental investigations of dynamic crack growth in various brittle materials have shown that the macro crack branching frequently originates from the nucleation of numerous microcracks ahead of a macro crack-tip. When the energy flux into the running crack-tip becomes sufficiently high, microscopic flaws in its vicinity start to grow and evolve into microcracks. With the increase of crack speed the number of nucleated microcracks also increases. The coalescence of the microcracks with the macro crack-tip can be interpreted as branching attempts which only at sufficiently high energy supply become successful and lead to macroscopically evolving crack branches. See Fig. 6 for an illustration.

3.2.1 Mathematical Aspects of Crack Branching

The “linear elastic fracture mechanics (LEFM)” provides a quantitative description of the motion of a single smooth crack in a linear elastic material. LEFM assumes that all the mechanical energy released during fracturing is dissipated at the crack tip. Defining \( G_f \) as the energy needed to create two crack surfaces of a unit area \( a \), the instantaneous crack growth velocity \( c \) is then selected by the balance between the energy flux and the
dissipation rate $G_f \cdot c$ which yields

$$G_f = \left(1 - \frac{c}{c_R}\right) K^0_I(t, l(t), 0) \quad (49)$$

where $c_R$ is the Rayleigh wave speed and $K^0_I(t, l(t), 0)$ is the dynamic stress intensity factor at a time $t$ for a stationary crack of length $l(t)$ which depends on the applied loading and the specimen geometry and completely characterizes the stress field in the vicinity of the crack front.

Equation (49) describes quantitatively the experimental results for dynamic brittle fracture at slow crack speeds. However, large discrepancies are observed in brittle materials at high crack speeds (Fineberg et al., 1991; Sharon et al., 1996). In particular, the measured maximum crack speeds lie in the range of $0.3c_R - 0.7c_R$ i.e. far smaller than the limiting speed $c_R$ predicted by the energy balance (49). It is observed that experiments start to depart from theory above a critical crack velocity $c_{crit} \approx 0.3c_R$ which is associated with the onset of microbranching instabilities; for $c > c_{crit}$ the crack motion becomes a multicrack state.

The question that arises here is that in brittle materials, in the absence of any rate-dependent dissipative mechanisms, $G_f$ should be independent of the crack speed $c$; why then does the crack not accelerate to the Rayleigh wave speed $c_R$? The answer to this was suggested by Ravi-Chandar and Knauss (1984b) that even in brittle materials, fracture proceeds with a significant process zone in which nucleation, growth and coalescence of microcracks occur and the dynamics of evolution of these processes and the microscopic path instabilities provide a rate and state-dependent character to the fracture energy; thus $G_f = G_f(c, K_I(t))$. Such an argument, as illustrated in Fig. 6, explains the discrepancies that arise by using (49) for higher crack velocities as the theoretical framework does not take into account the effect of dynamic instabilities associated with the growing crack.

### 3.2.2 Physical Aspects of Crack Branching

The number of crack branches emanating from a single propagating crack is found to be arbitrary with no physically sound criterion. A series of tests performed by Ravi-Chandar and Knauss (1984a,b) under identical conditions showed that while the instant of branching was nearly the same in all cases, the number of branches varied considerably. Hawong et al. (1987) observed in their experiments the relation between the biaxial loading, its ratio and number of successfully developing branches and the branching angles. However, experimentalists have observed the bifurcation or trifurcation of the growing cracks pre-
ceded by microbranching phenomena to be more occurrent than other cases which will be
discussed briefly here.

Symmetric Bifurcation: It is understood as a phenomena when a straight growing crack
branches into two symmetric macroscopic cracks such that the main crack ceases to grow
after the instance of branching. It is argued by Ramulu and Kobayashi (1985) that such
a phenomena is triggered by the coalescence of micro voids ahead of the growing crack
tip which nucleate under higher rate of loading and is attributed to a critical dynamic
stress intensity factor (SIF) at the crack tip. Rafiee et al. (2003) showed in their numeri-
cal investigations the correlation of dynamic SIF at the crack tip and the bifurcation
phenomena, with the value of critical SIF to be almost three times the initial value. The
first illustration in Fig. 7 depicts the experimental result of bifurcation of a growing crack

Symmetric Trifurcation: The third illustration in Fig. 7 shows an instance of trifurcation
obtained by experimental investigations of Murphy et al. (2006) using PMMA. Trifur-
cation is therefore understood as a phenomena where the main crack continues to grow
with the branching cracks after the instance of branching. In the numerical investigations
of Rafiee et al. (2003), symmetric trifurcation is found to take place at a higher ratio of
biaxial loading while the critical dynamic SIF at the crack tip may attain the same value
as in bifurcation.

Microbranching: Sharon et al. (1996) in their series of experimental investigations on
dynamic fracture in brittle materials like PMMA found out the phenomena of micro
branching which can be understood as numerous unsuccessful branching attempts before
the main branching event as shown in the second illustration of Fig. 7. The phenomena
of microbranching indicates the process of increased dissipation at higher supply of en-
ergy flux to the crack tip which is not sufficient to create macro crack surfaces. It was
also observed that microbranching phenomena caused fluctuations in the velocity of the
growing crack indicating the dynamic instabilities of the main crack.

3.2.3 Criteria for Crack Branching

Since the incipience of interest in crack branching phenomena, various theories have been
put forth to explain when a growing crack branches into multiple cracks. The most
prominent theories consider the distortion of crack tip stress field at a critical velocity or
the dynamic SIF at the crack tip attaining a critical value or the strain energy release
rate at the onset of branching and are briefly outlined here.

Critical Velocity Criterion: Long before solutions to elastodynamic crack problems were
available, Yoffe (1951) presented the steady-state solution for a crack of constant length,
moving along a straight line in an infinite two-dimensional medium under remote trac-
tions. Yoffe’s pre-branching analysis showed that the maximum circumferential stress,
$\sigma_{\theta\theta}$, exhibits two symmetrical maxima along the crack axis at a crack velocity of about
$c/c_d = 0.33$, where $c$ and $c_d$ are the crack- and dilatational wave velocities, respectively.
This critical velocity, which shifts the maximum $\sigma_{\theta\theta}$ orientation away from $\theta = 0$, was
suggested as promoting crack branching. Validation of the above theoretical solutions,
however, was stymied by the lack of detailed experimental data on the dynamic state of
stress in the vicinity of a moving crack tip.
Crack Branching in Dynamic Fracture


**Figure 7:** Illustrations of experimental results depicting the physical aspects of crack branching. The first result of Satoh (1999) shows an instance of bifurcation, the second result by Sharon et al. (1996) depicts the microbranching phenomena associated with high velocity cracks and the third result by Murphy et al. (2006) shows an instance of trifurcation.

**Critical Stress Intensity Criterion:** Attempts for analyzing the crack branching has sought the necessary condition for branching by comparing the stress states prior to and after branching. Since crack branching was observed at lower velocities, criticality of other fracture parameters like critical SIF or a critical stress in a region ahead of the crack tip must be considered.

Clark and Irwin (1966) concluded that crack branching occurs by advanced cracking, which requires a critical branching SIF $K_{ib}$. They also stated that the crack velocity approaches a limiting velocity before branching. Independently, Congleton and Petch (1967) proposed a critical SIF criterion for crack branching based on advanced cracking. A model of crack branching, which is based on advanced cracking of a Griffith crack, of length $2a$, located at a distance $r_c$, ahead of the propagating crack formed the micromechanic basis of their criterion. Further improvement of this crack branching model was made by Congleton (1973) where he derived a relation between the branching SIF $K_{ib}$ and the fracture toughness $K_{it}$. Experimental studies by Kobayashi and Mall (1978) showed that branching SIF is not a unique material property but depends on the loading conditions and experimental setup.

**The Energy Criterion:** The energy rate criterion requires crack branching when the strain energy release rate exceeds the energy which is dissipated by a single propagating crack. Based on experimental studies, Johnson and Holloway (1966, 1968) and Bansal (1977) attributed branching to the excessive supply rate of the driving energy to the propagating crack tip. Eshelby’s approach (Eshelby, 1970) to crack branching is also based on the energy balance at the onset of branching at a crack velocity larger than $0.3c_d$. By examining the experimental results for different materials with identical specimen geometry, Rossmanith and Irwin (1979) and Rossmanith (1980) indicated that heat loss had to be accounted for in predicting crack branching. Fuller et al. (1975) also observed the increase in temperature at the onset of crack branching.

We observe that the velocity of the growing crack plays an important role in all the aforementioned criterion and owing to the simplicity of such a criterion in terms of numerical implementation, we have adopted the critical velocity as the branching criterion in the numerical simulations presented in Chapter 5.
3.3 Challenges in Numerical Modeling

The numerical modeling of physical phenomena discussed in Sect. 3.2 is a challenging task as the inertia effects due to the dynamic nature of the problem need to be accounted for. In addition to this, crack surfaces should be allowed to propagate freely in the discretized domain.

3.3.1 Related Work

Miller et al. (1999) and Xu and Needleman (1994) in their endeavor for numerical modeling used large number of cohesive surfaces based on an intrinsic law which are distributed throughout the discretized domain to simulate crack branching in PMMA. Promising results were obtained with such an approach however the crack path is limited to the discretization of the domain and the phenomena of trifurcation is not simulated. However, the branching is detected by the numerical model itself and no additional criterion is needed.

An improvement on the above numerical model was made by Zhang et al. (2007) by using cohesive surfaces based on extrinsic law which is now able to simulate the microbranching instability. However the dependance of the crack path on the discretization of the domain and the use of a large number of triangular elements in the finite element mesh subsequently increasing the computational cost can be seen as the limitations of the above approach.

Another approach by Rafiee et al. (2003) uses a time-domain boundary element method and allows for curvilinear crack propagation where branching events are controlled by a criterion of critical mode I stress intensity factor while the propagation direction and growth rate of each branch are determined from the criterion of maximum circumferential stress.

Linder and Armero (2009) in an attempt to simulate crack branching in brittle materials developed a new finite element with embedded branching which is based on the embedded finite element method as discussed in Chapter 2. The branching cracks are accommodated in the element interior and are resolved by the correct representations of the kinematics of the finite element. This is accomplished by the identification of the proper separation modes characterizing these solutions and their incorporation in the discrete strain field of the element. A velocity based branching criterion is used to trigger branching of growing cracks. The embedded branching always takes place at a fixed angle of 90° and the primary crack does not grow beyond the branching element. Besides phenomena like microbranching is absent in the numerical simulations. However, good agreements with the results in Xu and Needleman (1994) are obtained.

3.3.2 A New Proposal

In view of the restrictions of the above approaches to model crack branching, the aim of this work is to use the computationally efficient embedded finite element method as the underlying principle to numerically model crack branching by allowing more than one strong discontinuity through a single finite element. As seen in the continuum problem of Fig. 8, the region surrounding the material point where branching is shown to occur
contains multiple crack surfaces. In the corresponding discretized problem, the instance of branching is captured by a single finite element containing multiple strong discontinuities. A crude approximation of such an instance of a finite element can be made by either deleting the contribution of that particular finite element to the global system of equations or by simply allowing a single strong discontinuity to develop in that particular finite element with branching discontinuities developing in the neighboring elements. However, the question about the state of branching as seen in Sect. 3.2.2 remains unanswered and is usually approximated by certain numerical restrictions.

The multilevel embedded finite element method proposed in the subsequent Chapter 4 treats the finite element where the branching criterion is met as a separate boundary value problem. The finite element is now adaptively discretized into a number of sub-elements in the incremental time step \([t_n, t_{n+1}]\) and a set of kinematic constraints is applied on the nodes of each edge. The effect of multiple crack surfaces is then modeled by allowing each sub-element to develop a strong discontinuity and the averaged properties associated with the main finite element are derived by matching the virtual energies at two levels of discretization.

The state of branching i.e. branching angle, bifurcation or trifurcation phenomena is then a result of the solution of the sub-boundary value problem. Such an approach allows to model the crack branching in a finite element in a precise manner with nearly no artificial restrictions to predict the state of branching.
4 A Multilevel Embedded Finite Element Method

In this chapter, a new computational tool, based on EFEM (Chapter 2), able to model multiple strong discontinuities through a single finite element by virtue of solving the boundary value problem at two separate levels will be presented.

4.1 Introduction

As discussed in Chapter 2, EFEM as a computational tool to model solids at failure has a limitation of treating only a single strong discontinuity through a finite element. The localization in the form of a single crack or a shear-band at any material point \( x \in B \) is treated by incorporating a strong discontinuity in the form of displacement jumps in the neighborhood \( B_x \subset B \) of the material point as shown in Chapter 2.3. Various failure phenomena like crack branching in brittle materials under dynamic loading, as discussed in Chapter 3, require resolution of multiple strong discontinuities through a single finite element to correctly predict the behavior of the failing solid.

Experiments performed by Kobayashi et al. (1974) on Homalite-100 sheets under dynamic loadings found the tendency of a single propagating crack to branch into multiple cracks with further theoretical and experimental framework presented in Kobayashi and Mall (1978) and Kobayashi and Ramulu (1981). Ravi-Chandar and Knauss (1984a) in their series of experimental investigations observed that the cracks propagate faster through the medium under higher rate of loading and for a critical value of stress intensity at the propagating crack tip multiple cracks were formed. The formation of crack branches is therefore understood as the phenomena of increased dissipation under higher rate of loading which Fineberg et al. (1991), Sharon et al. (1996), and Sharon and Fineberg (1996) confirmed in their experimental investigations where phenomena like micro-branching was observed to precede the main branching event. Various numerical methods, as outlined in Chapter 3.3.1, aimed to capture such a complex phenomenon only achieve qualitative comparisons to experimental results at best. The questions about the state of the crack branching at the crack-tip are approximated by certain numerical restrictions to achieve desirable outcomes.

In view of the above challenges, it is desirable to treat multiple strong discontinuities through a single finite element to represent the state of crack branching phenomena as precisely as possible. A new methodology, proposed in Chapter 3.3.2, will be presented in this chapter where the numerical modeling of multiple strong discontinuities will be shown in Sect. 4.3.

4.2 The Multilevel Concept

The technique of homogenization is widely used in case of non-linear heterogeneous materials to obtain an overall material response (Miehe and Koch, 2002; Miehe, 2003; Markovic et al., 2005; Sengupta et al., 2009), taking into consideration the micro-structural response, and is often denoted as micro-to-macro transition where the length scales are weakly coupled and are usually separated such that \( l_{\text{micro}}/l_{\text{macro}} \ll 1 \). The micro-structure is usually associated with every integration point in the numerical setting. In this work, the length
scales are strongly coupled and are finitely separated \((l_{\text{micro}}/l_{\text{macro}} < 1)\), hence no terminology of macro or micro quantities is used. A finite element model (Chapter 2) is used at both the length scales and the associated quantities are simply identified by their level of definition (level-1 or level-2).

The main goal in this section is to present the modeling of failure phenomena like crack branching in dynamic fracture of brittle materials by a proposed combination of the homogenization techniques with embedded finite element method as discussed in Chapter 2. For the standard initial boundary value problem (Chapter 2.2), we consider the bifurcation or trifurcation (Chapter 3.2.2) of a growing crack at a material point \(x \in \mathcal{B}\). Our aim is to incorporate the effect of the multiple strong discontinuities (e.g. three discontinuities characterized by surfaces \(\Gamma_{1}^x\), \(\Gamma_{2}^x\) and \(\Gamma_{3}^x\) for the case of trifurcation as illustrated in Fig. 9) in the neighborhood \(\mathcal{B}_x \subset \mathcal{B}\) of a material point \(x\).

4.2.1 The Definition of Two Levels

The region under consideration here is the neighborhood \(\mathcal{B}_x \subset \mathcal{B}\) of the material point \(x \in \mathcal{B}\) where the crack branching phenomena is occurring. In contrast to the approach presented in Chapter 2 where the failure was modeled by introducing a single strong discontinuity in the region \(\mathcal{B}_x\), we are interested here in computing the effect of multiple strong discontinuities in the region \(\mathcal{B}_x\).

In this scenario, we treat the region \(\mathcal{B}_x\) as a sub-domain defining the local problem at level-1 where the total displacement field \(\mathbf{u}_\mu(x, t)\) is acting. We now consider a sub-region \(\bar{\mathcal{B}}_{sx} \subset \mathcal{B}_x\) where a strong discontinuity \(\Gamma_{sx} \subset \mathbb{R}^{n_{\text{dim}} - 1}\) with unit normal \(\bar{n}\) in the form of a constant jump in the displacement field \([\bar{\mathbf{u}}_\mu]\) : \(\Gamma_{sx} \rightarrow \mathbb{R}^{n_{\text{dim}}}\) is incorporated. Sub-region \(\bar{\mathcal{B}}_{sx}\) defines the new local problem at level-2. See Fig. 9 for an illustration.

Accordingly we define the level-2 total displacement field \(\bar{\mathbf{u}}_\mu : \bar{\mathcal{B}}_{sx} \rightarrow \mathbb{R}^{n_{\text{dim}}},\) the total strain field \(\bar{\varepsilon}_\mu : \bar{\mathcal{B}}_{sx} \setminus \Gamma_{sx} \rightarrow \mathbb{R}^{n_{\text{dim}} \times n_{\text{dim}}}\) and the associated stress field \(\bar{\sigma} : \bar{\mathcal{B}}_{sx} \rightarrow \mathbb{R}^{n_{\text{dim}} \times n_{\text{dim}}}\) existing in kinematically admissible spaces such that

\[
\bar{\mathbf{u}}_\mu(\bar{x}, t) = \mathbf{u}(\bar{x}, t) + \bar{\mathbf{u}}([\bar{\mathbf{u}}_\mu], t) \quad \text{and} \quad \bar{\varepsilon}_\mu = \bar{\varepsilon}(\mathbf{u}) + \bar{\varepsilon}(\bar{\mathbf{u}}) \quad (50)
\]
in terms of the smoothly continuous part, and the discontinuous part for the local coordinate \( \bar{x} \in \mathcal{B}_x \) and time \( t \in \mathbb{R} \). Refer Sect. 2.3.1 for details. The level-2 stresses are related to level-2 strains, in regions where no discontinuity is present, by the same constitutive relation as shown in (27).

The local problem at level-1 now defines a new boundary value problem which is referred to as sub-boundary value problem (SBVP) here; in addition to the global initial boundary value problem (GBVP). The static equilibrium of the sub-domain is assumed as shown in Sect. 4.2.2 with a kinematic constraint applied on its boundary as shown in Sect. 4.2.3. The coupling between the two levels is then achieved by matching their virtual energies due to admissible variations in an incremental time step \([t_n, t_{n+1}]\).

In the discretized domain \( \mathcal{B}^h \), each finite element \( \mathcal{B}^h_e \) where the crack branching criterion is met represents the sub-domain, denoted by \( \mathcal{B}^h_e \), and representing the level-1 of local problem, which is adaptively discretized into a number of sub-elements \( \mathcal{B}^h_{se} \) representing the level-2 of local problem. The strong discontinuity \( \mathcal{\Gamma}^h_{se} \) is now incorporated as a jump \([\bar{u}^h_{\mu}]\) in the displacement field of the sub-elements \( \mathcal{B}^h_{se} \) and the averaged internal force vector and stiffness matrix are obtained by solving the sub-boundary value problem associated with the sub-domain \( \mathcal{B}^h_e \). See Fig. 10 for an illustration.

### 4.2.2 The Static Equilibrium At Level-1

The sub-domain is loaded by tractions from the bulk \( \mathcal{B} \backslash \mathcal{B}_x \) only on the boundary \( \partial \mathcal{B}_x \) and its dynamic equilibrium state satisfies the balance of linear momentum

\[
\int_{\mathcal{B}_x} \rho (b_x - \bar{\dot{u}}_{\mu}) \, dV + \int_{\partial \mathcal{B}_x} t_x \, dA = 0 \tag{51}
\]

for the body force \( b_x \) and traction \( t_x \) related to \( \mathcal{B}_x \). The classical homogenization theory defines the overall constitutive response of the sub-domain based on a static equilibrium state which neglects the body and inertial forces. Therefore from (51), we obtain the equations governing the equilibrium state of the sub-domain as

\[
\int_{\partial \mathcal{B}_x} t_x \, dA = 0 \quad \text{and} \quad \int_{\partial \mathcal{B}_x} \bar{x} \times t_x \, dA = 0 \tag{52}
\]

where the second equation in (52) represents the balance of angular momentum. Using Cauchy’s theorem to express traction in terms of stress, we can recast (52) into a well known local form as

\[
\text{div}[\bar{\sigma}] = 0 \quad \text{and} \quad \bar{\sigma} = \bar{\sigma}^T \tag{53}
\]

where the first equation in (53) represents the local form of balance of linear momentum for the static equilibrium and second equation represents the symmetry of Cauchy stress introduced in the local problem at level-2. The weak form of (53) is obtained in the similar manner as shown in Chapter 2.2.1 as

\[
\int_{\mathcal{B}_x} \bar{\sigma} : \nabla(\delta \bar{u}_{\mu}) \, dV = 0 \tag{54}
\]
for admissible variations $\delta \tilde{u}_\mu \in \tilde{V}$ such that $\tilde{V} = \{ \delta \tilde{u}_\mu | \delta \tilde{u}_\mu \in H^1, \delta \tilde{u}_\mu(\partial B_x) = 0 \}$ which is solved numerically by using the embedded finite element method (Chapter 2), satisfying the kinematic constraints as discussed in Sect. 4.2.3, to obtain the level-2 displacement field $\tilde{u}_\mu$.

### 4.2.3 The Coupling Between Two Levels

Three types of classical boundary constraints can be applied on the sub-domain to obtain the averaged properties namely (i) linear deformation, (ii) uniform tractions and, (iii) periodic displacements. We will focus on linear deformations initially applied on the boundary of the sub-domain as shown below.

The displacement field of the boundary of sub-domain must match the global field acting on it. This can be assumed by a simple linear deformation constraint written here in a generalized form as

$$\tilde{u}_\mu = \phi(\bar{x}, u_\mu) \quad \text{on} \quad \partial B_x$$

where $\phi \in \mathbb{R}^{n_{dim}}$ is a linear function defining the constraint for e.g. $\tilde{u}_\mu = u_\mu$ when no discontinuities have crossed the sub-domain boundary. The constraint $\phi$ is modified accordingly in case of a discontinuity through the sub-domain boundary as shown in Sect. 4.3.5. The transition from level-2 to level-1 can be obtained by the virtual work balance between the two levels due to the respective admissible variations shown here as

$$\delta W(\delta u_\mu) = \delta \tilde{W}(\delta \tilde{u}_\mu) = \int_{B_x} \delta \tilde{w}(\delta \tilde{u}_\mu).$$

where $\delta W$ is the virtual work done at level-1 due to variations $\delta u_\mu$ and $\delta \tilde{W}$ is the total virtual work done over the sub-domain $B_x$ in terms of virtual work $\delta \tilde{w}$ done at level-2.

The coupling terms are derived finally at the local equilibrium of the sub-boundary value problem which can be stated as:

**Given**: $u_\mu$, $\tilde{u}_\mu \in \partial B_x$ at $t_n$.

**Find**: $\tilde{u}_\mu \in B_x \setminus \partial B_x$ such that for any admissible variation $\delta \tilde{u}_\mu$, $\delta \tilde{W} = 0$. 

---

**Figure 10**: Illustration of the concept of multilevel embedded finite element method shown in the discretized global domain $B^h$ where an element $B^h_e \subset B^h$ represents a sub-domain $\bar{B}^h_e$ which is discretized in a time step $[t_n, t_{n+1}]$ into $n_{sele}$ sub-elements $B^h_{se}$, where each sub-element is capable of developing a strong discontinuity $\bar{\Gamma}^h_{se}$. 

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4.3 The Finite Element Modeling

In the discretized domain $B^h$, the finite element $B^h_e$ where the crack branching criterion has been met, denotes the sub-domain $(\overline{B}^h_e)$ which is consequently adaptively discretized into a number of finite elements $\overline{B}^h_{se}$, called as sub-elements at a time step $t_n \in [0, T]$ (Fig. 10). Each sub-element $\overline{B}^h_{se}$ has the capability of developing a strong discontinuity $\Gamma^h_{se}$ with unit normal $\overline{n}$ and tangent $\overline{m}$ vectors and the associated jump $[\overline{u}^h_{\mu}]$ in the displacement field. The finite element approximation of the displacement field and the strain field associated with $\overline{B}^h_{se}$ is done in the similar manner as shown in Chapter 2.3.2. The main goal here is the computation of the averaged level-1 internal force vector $\overline{R} \in \overline{B}^h_e$ and stiffness matrix $\overline{K} \in \overline{B}^h_e$. This is done by obtaining the solution of (54) using the EFEM and then enforcing the constraint (55) for the coupling between two levels using (56).

4.3.1 Adaptive Discretization of the sub-domain

The discretization of the sub-domain is not performed a priori as the finite element $B^h_e$ where the crack branching criterion is met depends on the state of dynamic loading. It, therefore, poses an additional challenge in the implementation of the multilevel embedded finite element method where a sub-domain must be discretized during the runtime of the simulation.

This is achieved by an algorithm which detects the boundary of the sub-domain and creates pseudo nodal positions on the boundary as well as in the interior of sub-domain with the same degrees of freedom as for global nodal positions. The newly created nodal positions, so-called pseudo nodes, do not contribute to the global set of discrete equations and are needed to create sub-elements $\overline{B}^h_{se}$. In the two-dimensional setting, only four-node quadrilateral sub-elements are used in the numerical simulations presented in Chapter 5.

The number of sub-elements $n_{sele}$ created depend on the development of the strong discontinuity through the sub-elements as only a single discontinuity is allowed to propagate in each sub-element. In case this condition is violated, the discretization is refined till a sub-element represents only a single discontinuity. See Fig. 11 for an illustration.

The pivotal concept of multilevel embedded finite element method lies in the idea of introducing a single discontinuity in the form of a jump in the displacement field of a sub-element where the application of the classical embedded finite element method becomes feasible.

4.3.2 The Partition of Non-Linear Algebraic Equations

The nodes of the discretized element $\overline{B}^h_e$ are partitioned into the nodes $\overline{x}_b \in \partial \overline{B}^h_e$ lying on the boundary and nodes $\overline{x}_i \in \overline{B}^h_e \\setminus \partial \overline{B}^h_e$ lying in the interior of the sub-domain. We also partition the assembled internal force vector and stiffness matrix obtained by the application of the EFEM at level-2 accordingly as

$$\overline{x} = \begin{bmatrix} \overline{x}_i \\ \overline{x}_b \end{bmatrix}, \quad \overline{R}_e = \begin{bmatrix} \overline{R}_i \\ \overline{R}_b \end{bmatrix}, \quad \overline{K}_e = \begin{bmatrix} \overline{K}_{ii} & \overline{K}_{ib} \\ \overline{K}_{bi} & \overline{K}_{bb} \end{bmatrix}$$

(57)

for the arrays $\overline{R}_e \subset \mathbb{R}^{\overline{n}_{dof}}$ and $\overline{K}_e \subset \mathbb{R}^{\overline{n}_{dof} \times \overline{n}_{dof}}$, where $\overline{n}_{dof} = n_{\text{dim}} \cdot \overline{n}_{\text{node}}$ is the total
number of degrees of freedom and $\bar{n}_{\text{node}}$ is the total number of nodes in $\overline{B}_e^h$, written as

$$\bar{R}_e = \sum_{se=1}^{\overline{n}_{\text{sele}}} \bar{R}_{eff}^{se} \quad \text{and} \quad \bar{K}_e = \sum_{se=1}^{\overline{n}_{\text{sele}}} \bar{K}_{eff}^{se} \in \overline{B}_e^h$$

(58)

in terms of contributions $\bar{R}_{eff}^{se}$ and $\bar{K}_{eff}^{se}$ from $n_{sele}$ sub-elements obtained by using the embedded finite element method at level-2 as shown in Sect. 4.4.1.

### 4.3.3 Generalized Boundary Constraint

Starting with the constraint (55) of linear deformation on the boundary of the sub-domain, the nodal displacement for each node $k$ on the discretized boundary $\partial \overline{B}_e^h$ is given as

$$\bar{d}_k(\bar{x}_k) = \phi_k^h(\bar{x}_k, d_e).$$

(59)

Here $\phi_k^h(\bar{x}_k, d_e) \in \mathbb{R}^{n_{\text{dim}}}$ is a linear deformation constraint applied to the node $k$ at position $\bar{x}_k$ and $d_e \in \mathbb{R}^{n_{\text{dof}}}$ is the nodal displacements associated with $\overline{B}_e^h$. An example of $\phi_k^h$ for a Q1 element with no discontinuity across the element edge is

$$\phi_k^h = N(\bar{x}_k) d_e = \begin{bmatrix} N_1^k \cdot d_{1x} + N_2^k \cdot d_{2x} + N_3^k \cdot d_{3x} + N_4^k \cdot d_{4x} \\ N_1^k \cdot d_{1y} + N_2^k \cdot d_{2y} + N_3^k \cdot d_{3y} + N_4^k \cdot d_{4y} \end{bmatrix}$$

(60)

where $N(\bar{x}_k) \in \mathbb{R}^{h}$ is the shape function computed at position $\bar{x}_k$. For all $\bar{n}_{\text{node}}^B$ nodes on the boundary $\partial \overline{B}_e^B$ of the sub-domain, we define a matrix $\Phi^h \in \mathbb{R}^{\bar{n}_{\text{dof}}^B}$ such that

$$\Phi^h := \begin{bmatrix} \phi_1^h & \phi_2^h & \phi_3^h & \cdots & \phi_{\bar{n}_{\text{node}}^B}^h \end{bmatrix} \text{Transpose}$$

(61)

where $\bar{n}_{\text{dof}}^B = n_{\text{dim}} \cdot \bar{n}_{\text{node}}^B$. Using (61), one can write the constraint for all the nodes associated with the boundary as

$$\overline{d}_b = \Phi^h(d_e).$$

(62)
Figure 12: Illustration of the discrete local problem at level-1 represented by the discretized finite element $\bar{\mathcal{B}}^h_e$ where multiple discontinuities e.g. $\Gamma^1_e$, $\Gamma^2_e$ and $\Gamma^3_e$ may develop. Sub-element $\bar{\mathcal{B}}^h_{se}$ represents the level-2 of the local problem where a discontinuity $\bar{\Gamma}^h_{se}$ is incorporated by introducing a jump $[\bar{u}^h_\mu]$ in the displacement field of $\bar{\mathcal{B}}^h_{se}$.

4.3.4 Balance of Virtual Work

As seen in Sect. 4.2.3, virtual works at the local equilibrium in the incremental time step $[t_n, t_{n+1}]$ are matched to derive coupling terms. In the discretized domain for an element $\mathcal{B}^h_e$, this can be simply written as

$$ \delta \mathbf{d}^T_e \mathbf{R}_e = \delta \mathbf{d}^T_i \mathbf{R}_i $$

$$ \Rightarrow \delta \mathbf{d}^T_e \mathbf{R}_e = \delta \mathbf{d}^T_e \mathbf{R}_b + \delta \mathbf{d}^T_i \mathbf{R}_i $$

(63)

where the use of partitioning in (57) is made in (63). The finite element solution for the sub-boundary value problem result in $\mathbf{R}_b = 0$. Taking the variation of (62) and substituting in (63) yields

$$ \delta \mathbf{d}^T_e \mathbf{R}_e = \delta \mathbf{d}^T_e \mathbf{R}_b $$

$$ \Rightarrow \delta \mathbf{d}^T_e \mathbf{R}_e = \delta \mathbf{d}^T_e \left( \frac{\partial \Phi^h}{\partial \mathbf{d}_e} \right)^T \mathbf{R}_b. $$

(64)

For all admissible variations of nodal displacements $\delta \mathbf{d}_e \neq 0$, we obtain the coupling term for the internal force vector in terms of internal forces $\mathbf{R}_b \in \mathbb{R}^{n_{B_dof}}$ of the nodes lying on the boundary of the discretized sub-domain $\bar{\mathcal{B}}^h_e$. Also $\frac{\partial \Phi^h}{\partial \mathbf{d}_e} =: \mathbf{G} \in \mathbb{R}^{n_{B_dof} \times n_{dof}}$ acts as a projection tensor projecting the values from nodes at level-2 to nodes at level-1 depending upon the kinematic boundary constraint $\Phi^h$. Therefore from (64), we get

$$ \mathbf{R}_e = \mathbf{G}^T \mathbf{R}_b. $$

(65)

The projection tensor $\mathbf{G}$ is described for each nodal position on the boundary of the sub-domain as

$$ \mathbf{G} = [G_{\bar{1}} G_{\bar{2}} G_{\bar{3}} ... G_{\bar{n}_B^{node}}]^\text{Transpose} $$

(66)

where for each nodal position $k = \bar{1}, ..., \bar{n}_B^{node}$, one writes

$$ G_{\bar{k}} = \frac{\partial \Phi^h_{\bar{k}}}{\partial \mathbf{d}_e} =: \frac{\partial \Phi_i}{\partial \mathbf{d}_j} e_i \otimes e_j \in \mathbb{R}^{n_{dim} \times n_{dof}}. $$

(67)
**Stiffness Matrix Computation:** For the finite element solution, consistent linearization is performed at both levels to use Newton iterative solver. Formally, the linearization is given by

\[ \delta \mathbf{d}_e^T \frac{\partial \mathbf{R}_e}{\partial \mathbf{d}_e} \mathbf{d}_e = \delta \mathbf{d}_e^T \frac{\partial \mathbf{R}_e}{\partial \mathbf{d}} \mathbf{d}. \]  

(68)

The sensitivity of \( \mathbf{R}_e \) with respect to \( \mathbf{d}_e \) is the stiffness matrix \( \mathbf{K}_e \in \mathbb{R}^{n_{\text{dof}} \times n_{\text{dof}}} \) at level-1. Similarly the sensitivity of \( \bar{\mathbf{R}}_e \) with respect to \( \bar{\mathbf{d}} \) is the assembled stiffness matrix \( \bar{\mathbf{K}}_e \in \mathbb{R}^{\bar{n}_{\text{dof}} \times \bar{n}_{\text{dof}}} \) of level-2. Therefore (68) can be written as

\[ \delta \mathbf{d}_e^T \mathbf{K}_e \mathbf{d} = \delta \bar{\mathbf{d}}^T \bar{\mathbf{K}}_e \bar{\mathbf{d}}. \]  

(69)

Again making use of the partition in (57), we can rewrite (69) as

\[ \delta \mathbf{d}_e^T \begin{bmatrix} 0 & \mathbf{K}_e \mathbf{d}_e \\ \mathbf{K}_e \mathbf{d}_e \end{bmatrix} = \left[ \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ib} \\ \mathbf{K}_{bi} & \mathbf{K}_{bb} \end{bmatrix} \right] \begin{bmatrix} \mathbf{d}_i \\ \mathbf{d}_b \end{bmatrix}. \]  

(70)

For an arbitrary set of non-zero variations, we can write the above equation as

\[ \begin{bmatrix} 0 & \mathbf{K}_e \mathbf{d}_e \\ \mathbf{K}_e \mathbf{d}_e \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ib} \\ \mathbf{K}_{bi} & \mathbf{K}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{d}_i \\ \mathbf{d}_b \end{bmatrix}. \]  

(71)

(72)

and static condensation of \( \mathbf{d}_e = -\mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} \mathbf{d}_b \) results into

\[ \mathbf{K}_e \mathbf{d}_e = \mathbf{K}_{bi} \left[ -\mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} \mathbf{d}_b \right] + \mathbf{K}_{bb} \mathbf{d}_b \]

\[ = -\mathbf{K}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} \left( \frac{\partial \Phi^h}{\partial \mathbf{d}_e} \right) \mathbf{d}_e + \mathbf{K}_{bb} \left( \frac{\partial \Phi^h}{\partial \mathbf{d}_e} \right) \mathbf{d}_e \]

\[ \Rightarrow \mathbf{K}_e = \begin{bmatrix} \frac{\partial \Phi^h}{\partial \mathbf{d}_e} \end{bmatrix}^T \left[ \mathbf{K}_{bb} - \mathbf{K}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} \right] \begin{bmatrix} \frac{\partial \Phi^h}{\partial \mathbf{d}_e} \end{bmatrix}. \]  

(73)

Equation (62) is used in (73) to replace level-2 nodal increments. We also observe that \( \mathbf{K}_{bb} - \mathbf{K}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} =: \mathbf{K}_{bb} \in \mathbb{R}^{n_{\text{dof}} \times n_{\text{dof}}} \) represents the condensed stiffness matrix associated with the boundary nodes of the subdomain \( \bar{\mathcal{B}}_e^h \). Equation (73) can therefore be written in a concise manner as

\[ \mathbf{K}_e = \mathbf{G}^T \mathbf{K}_{bb} \mathbf{G}. \]  

(74)

### 4.3.5 Remarks on the Boundary Constraint

It is observed that the linear deformation constraint on the boundary of the sub-domain becomes inconsistent as soon as a strong discontinuity \( \bar{\Gamma}_{se} \subset \bar{\mathcal{B}}_{se}^h \) crosses the element edge.
The inconsistency arises in the form of spurious stresses in the subelements resulting in an incorrect definition of the averaged quantities in (65) and (74).

To obtain the correct representation of the level-1 internal force vector and stiffness matrix, the linear deformation constraint (60) for each nodal position is modified. In Fig. 13, four different cases of boundary constraint modifications over a finite element edge are illustrated. Here we consider the discretized edge of an element $\bar{B}^h_e$, with level-2 nodal positions denoted by $\bar{1}, \bar{2}, \ldots, \bar{6}$, which is connected to level-1 nodes $1, 2, 3, 4$ of the finite element through the given constraint $\phi^h$ for e.g. Case I represents the standard linear deformation constraint when no strong discontinuity crosses the element edge. The nodal displacement at each position is given by equation (60) as shown in Sect. 4.3.3.

Case I: When no strong discontinuity crosses the element edge.

For the linear deformation constraint $\phi^h$ at each nodal position as shown in (60), we can write the projection tensor $\mathbf{G}_k$ for each node in this case (Fig 13 - I) as

$$
\mathbf{G}_k = \begin{bmatrix}
N^1_k & 0 & N^2_k & 0 & N^3_k & 0 & N^4_k & 0 \\
0 & N^1_k & 0 & N^2_k & 0 & N^3_k & 0 & N^4_k
\end{bmatrix}
$$

(75)

where $k = \bar{1}, \ldots, \bar{6}$ for the element edge under consideration.

Case II: When a strong discontinuity crosses the element edge, say at position $x_{\Gamma}$ with normal $\bar{n}$.

The constraint at a nodal position $x_k$ on the element edge can be obtained as

$$
\phi^h_k = \begin{cases}
d_1 & \text{if } (x_k - x_{\Gamma}) \cdot \bar{n} < 0 \\
d_2 & \text{if } (x_k - x_{\Gamma}) \cdot \bar{n} > 0
\end{cases}
$$

(76)

for a fully softened SD $\bar{\Gamma}^h_{se}$ through the element edge. Therefore the constraint vector for nodes $\bar{1}, \ldots, \bar{6}$ of the illustrated edge (Fig 13 - II) results into (77).

Such a constraint formulation is motivated by fulfilling the condition of total opening $j = [\llbracket \bar{u}_{\Gamma} \rrbracket] = |d_1 - d_2|$ at $x_{\Gamma}$ for the fully softened case considered in the numerical simulations presented in Chapter 5.
The constraint vector for this case is given by
\[
\begin{bmatrix}
\phi^h_1 \\
\phi^h_2 \\
\phi^h_3 \\
\phi^h_4 \\
\phi^h_5 \\
\phi^h_6 \\
\end{bmatrix} =
\begin{bmatrix}
d_1 \\
d_1 \\
d_2 \\
d_2 \\
d_2 \\
d_2 \\
\end{bmatrix}.
\]
(77)

From (77), the projection tensor associated with the nodes of the given element edge can be written as
\[
\mathbf{G}_1 = \mathbf{G}_2 = \mathbf{G}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]
\[
\mathbf{G}_4 = \mathbf{G}_5 = \mathbf{G}_6 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]
and \( \mathbf{G}_k \) at a nodal position on any other edge is found in the same manner.

Case III: When two strong discontinuities cross the element edge, say at \( x_{\Gamma_1} \) and \( x_{\Gamma_2} \) with normals \( \vec{n}_1 \) and \( \vec{n}_2 \) respectively.

The constraint at a nodal position \( \vec{x}_k \) on the element edge can be obtained as
\[
\phi^h_k = \begin{cases} 
\mathbf{d}_1 & \text{if } (\vec{x}_k - x_{\Gamma_i}) \cdot \vec{n}_j < 0 \\
\mathbf{d}_2 & \text{if } (\vec{x}_k - x_{\Gamma_i}) \cdot \vec{n}_j > 0 \\
0 & \text{else}
\end{cases}
\]
(78)

for \( i, j = 1, 2 \) i.e. for a nodal position at \( \vec{x}_1 \), \( \phi^h_1 = \mathbf{d}_1 \) if \( (\vec{x}_1 - x_{\Gamma_1}) \cdot \vec{n}_1 < 0 \), \( (\vec{x}_1 - x_{\Gamma_2}) \cdot \vec{n}_2 < 0 \), \( (\vec{x}_1 - x_{\Gamma_2}) \cdot \vec{n}_1 < 0 \) and \( (\vec{x}_1 - x_{\Gamma_2}) \cdot \vec{n}_2 < 0 \). Both the SDs are assumed to open with a certain jump for the numerical convenience and distinction from case II. Hence the constraint vector for nodes \( \vec{x}_1, ..., \vec{x}_6 \) of the illustrated edge (Fig 13 - III) results into
\[
\begin{bmatrix}
\phi^h_1 \\
\phi^h_2 \\
\phi^h_3 \\
\phi^h_4 \\
\phi^h_5 \\
\phi^h_6 \\
\end{bmatrix} =
\begin{bmatrix}
d_1 \\
d_1 \\
0 \\
0 \\
d_2 \\
d_2 \\
\end{bmatrix}.
\]
(79)

Again the requirement of total opening \( j = |[\mathbf{u}_{\Gamma_1}] + [\mathbf{u}_{\Gamma_2}]| = |\mathbf{d}_1 - \mathbf{d}_2| \) at \( x_{\Gamma_1} \) and \( x_{\Gamma_2} \) is fulfilled. From (79), the projection tensor associated with the nodes of the given element edge can be written as
\[
\mathbf{G}_1 = \mathbf{G}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]
\[
\mathbf{G}_3 = \mathbf{G}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]
\[
\mathbf{G}_5 = \mathbf{G}_6 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]

and again the same formulation is valid for other element edges with the same condition.
Case IV: When three strong discontinuities cross the element edge, say at \( x_{\Gamma_1}, x_{\Gamma_2} \) and \( x_{\Gamma_3} \) with normals \( \bar{n}_1, \bar{n}_2 \) and \( \bar{n}_3 \) respectively.

The complexity of modifying the boundary constraints increases with the number of discontinuities through the element edge. In the case of three SDs, the constraint at a nodal position \( \bar{x}_k \) on the element edge can be obtained as

\[
\phi^h_k = \begin{cases} 
    d_1 & \text{if } (\bar{x}_k - x_{\Gamma_i}) \cdot \bar{n}_j < 0 \\
    d_2 & \text{if } (\bar{x}_k - x_{\Gamma_i}) \cdot \bar{n}_j > 0 \\
    q_k & \text{else}
\end{cases}
\]

(80)

for \( i, j = 1, 2, 3 \) and \( q_k = \frac{a_k d_1 + b_k d_2}{n} \) where \( n \) is the number of sub-elements along the element edge. The constants \( a_k \) and \( b_k \) at a nodal position are derived from the topology of the three discontinuities through the element edge. For the case under consideration (Fig 13 - IV), the constraint vector for the element edge results into

\[
\begin{bmatrix}
\phi^h_1 \\
\phi^h_2 \\
\phi^h_3 \\
\phi^h_4 \\
\phi^h_5 \\
\phi^h_6
\end{bmatrix} = \begin{bmatrix}
d_1 \\
d_1 \\
(2d_1 + d_2)/3 \\
(d_1 + 2d_2)/3 \\
(d_1 + 2d_2)/3 \\
d_2
\end{bmatrix}
\]

(81)

where again the requirement of total opening \( j = |[\bar{u}_{\Gamma_1}] + [\bar{u}_{\Gamma_2}] + [\bar{u}_{\Gamma_3}]| = |d_1 - d_2| \) is fulfilled by equal jumps \( |[\bar{u}_{\Gamma_1}]| = |[\bar{u}_{\Gamma_2}]| = |[\bar{u}_{\Gamma_3}]| = j/3 \) at \( x_{\Gamma_1}, x_{\Gamma_2} \) and \( x_{\Gamma_3} \) for numerical convenience and for distinction from case III. It is to be noted that such an assumption is valid only for the case of fully softened SDs in the sub-domain \( B^h \). From (81), the projection tensor associated with the nodes of the given edge can be written as

\[
\mathbf{G}_1 = \mathbf{G}_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathbf{G}_3 = \begin{bmatrix}
2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 \\
0 & 2/3 & 0 & 1/3 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathbf{G}_4 = \mathbf{G}_5 = \begin{bmatrix}
1/3 & 0 & 2/3 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/3 & 0 & 2/3 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathbf{G}_6 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The cases I-IV completely generalize the boundary constraints applied to a sub-domain which are consistent when SDs are developed across the element edges. The projection tensor \( \mathbf{G} \) in (66) can be any combination of the aforementioned cases which depends upon the solution of SBVP.

**Remark:**

For number of sub-elements \( n = 3 \) along the element edge, the constraint vector (81) of case IV is similar to constraint vector (60) of case I for a regular and structured mesh.
4.4 Numerical Implementation

The formulation presented in Sect 4.3 supplements the current embedded finite element code. In the discretized domain $B^h$, a SBVP needs to be solved for a finite element $B^h_e$ where the crack branching criterion has been met resulting in the definition of the averaged quantities associated with $B^h_e$ as shown in Sect. 4.3.4. The SBVP is solved in every Newton iteration $k$ of an incremental time step $[t_n, t_{n+1}]$ of the GBVP.

4.4.1 Solution of the Sub-Boundary Value Problem

For the weak form (54) representing the static equilibrium of the sub-domain $(\bar{\Gamma}^h_{se})$, we compute the effect of a strong discontinuity $B^h_e$ by introducing the jump in the displacement field $\Delta d$ in the form of an effective residual and stiffness (48) of the element $B^h_e \subset \bar{\mathcal{E}}_{\text{disc}}$, and $\mathbf{R}_{se}^{\text{eff}} = \mathbf{R}_{se}^{\text{enh}} - \mathbf{K}_{se}^{\text{eff}} \mathbf{R}_{se}^{\text{enh}}$ (84) in terms of enhanced arrays $\mathbf{K}_{se}^{\text{enh}}$, $\mathbf{K}_{se}^{\text{eff}}$ and $\mathbf{K}_{se}^{\text{enh}}$ associated with each sub-element with a discontinuity. The quantities in (84) are subsequently assembled to represent the internal force vector and stiffness matrix associated with the $B^h_e$ as seen in (58).

4.4.2 Effective Residual and Stiffness

In the embedded finite element procedure, we computed the effect of a strong discontinuity $\Gamma^h_e \subset B^h_e$ in the form of an effective residual and stiffness (48) of the element $B^h_e \subset \bar{\mathcal{E}}_{\text{disc}}$ which is assembled in (47) to solve for the global displacement field $\Delta d$. The modeling of crack branching results in the development of multiple strong discontinuities within a single finite element $B^h_e \subset B^h$. As mentioned earlier in the Sect 4.2, the multiple strong discontinuities are incorporated within a finite element $B^h_e$ by introducing the jump in the displacement field of the sub-elements $\bar{B}^h_{se} \subset \bar{B}^h_e$. That means no strong discontinuity is
present in the displacement field of $B^h_e$ and, by following the standard embedded finite element procedure, no enhanced arrays $K_{de}^e$, $K_{e}^{d\xi}$ and $K_{e}^{d\xi}$ associated with $B^h_e$ needs to be computed. Therefore for an element with multiple strong discontinuities, (48) can be written as

$$K_{eff}^e = \tilde{K}_{dd}^e \quad \text{and} \quad R_{eff}^e = R^e$$ (85)

The multilevel embedded finite element method, proposed in this chapter, computes the effect of multiple strong discontinuities within an element $B^h_e$ by solving the SBVP (54) subjected to the kinematic constraint (55) which is solved for a local equilibrium to obtain the averaged internal force vector $R_e$ in (65) and stiffness matrix $K_e$ in (74) as shown in Sect. 4.3.4. These averaged quantities are now directly used in (85) to obtain

$$\tilde{K}_{dd}^e = K_e + \frac{1}{\beta\Delta t^2}M_e^e \quad \text{and} \quad R^e = R_e + M_e a^e$$ (86)

the effective stiffness and effective residual associated with the finite element $B^h_e$ with multiple strong discontinuities. Such a formulation maintains the robustness and structure of the embedded finite element code as the final system of discrete governing equations in (47) remain unchanged. The flow chart below summarizes the implementation of the multilevel embedded finite element method.

It is shown that for every $k^{th}$ Newton iteration of the GBVP, a SBVP is solved for an element $B^h_e$ with multiple strong discontinuities (MSDs) to obtain the effective stiffness and residual which are then assembled into the global arrays $R_e$ and $K_e$ to solve for global and local degrees of freedom iteratively.
Remarks on the stability of the sub-domain: The cracks in the form of strong discontinuities are introduced at level-2 as the jumps in the displacement field which are detected by the loss of ellipticity criterion (critical stress) as discussed in Chapter 2. Such an approach does not require the regularization of the model and hence no stability analysis is implemented.
5 Representative Numerical Simulations

This chapter will present the simulation results obtained by using the multilevel embedded finite element method as discussed in chapter 4 and shown here in Sect. 5.4 and compare the results obtained by using the classical embedded finite element method as discussed in chapter 2 and shown here in Sect. 5.3.

5.1 Element Tests

We consider an element test for the evaluation of locking free properties of the proposed method and the correct choice of parameters regarding the traction-separation law to be employed in the sub-boundary value problem.

![Figure 14](image.png)

**Figure 14**: A Q1 element subjected to displacement loads and constraints is shown on the left which is discretized during the runtime into 9 sub-elements with linear deformation applied on the boundary nodes as shown on the right. Each sub-element develops a strong discontinuity (shown in red) depending upon its stress state.

5.1.1 Problem Statement

A $200 \times 200 \times 1 \, mm^3$ block represented by a single 4-node quadrilateral (Q1) element as shown in the left of Fig. 14 is considered, under plane stress conditions, following the linear elastic relation (27) for the bulk with a Young’s modulus of $E = 30 \, GPa$ and Poisson ratio of $\nu = 0.2$. The assumed tensile strength is $f_t = 3 \, MPa$. A piece-wise linear softening law is considered in the normal and tangential directions along the crack surface as

$$
t_{\Gamma_n} = \max\{0, f_{t_n} + S [u^h_{\mu_n}]\}, \quad t_{\Gamma_m} = \frac{[u^h_{\mu_m}]}{[u^h_{\mu_m}]} \cdot \max\{0, f_{t_m} + S [u^h_{\mu_m}]\}
$$

where the traction $t_{\Gamma_n}$ in normal direction only depends on normal jumps $[u^h_{\mu_n}]$ and the traction in tangential direction $t_{\Gamma_m}$ only depends on the tangential jump $[u^h_{\mu_m}]$ for the softening modulus $S = -45 \, MPa/mm$ corresponding to the fracture energy $G_f = f_t^2 / (2|S|) = 0.1 \, N/mm$. The maximum strength in normal and tangential direction is assumed to be equal to the maximum tensile strength ($f_{t_n} = f_{t_m} = f_t$).

The block is subjected to the boundary conditions as shown in the left of Fig. 14 where a displacement of $\pm0.08 \, mm$ is applied on left corner nodes and $\pm0.05 \, mm$ on the right.
corner nodes of the top and bottom edges in the vertical direction. In addition to this, a horizontal displacement of 0.01 mm is also applied on the right corner nodes. Such a boundary condition is motivated by the actual deformation state of an element where the crack branching criterion has been met in the numerical simulation shown later in Sect. 5.3 and Sect. 5.4.

Under the quasi-static equilibrium condition, as soon as the stress state of the block reaches $f_t$, it is adaptively discretized into 9 sub-elements as shown in the right of Fig. 14. The nodal positions on the boundary of the discretized element is computed as per the constraint shown in chapter 4.2.3. Following the key argument of the multilevel embedded finite element method, each sub-element has the possibility of developing a strong discontinuity, therefore, depending upon the stress state of each sub-element, a strong discontinuity develops as shown by the red lines in the right of Fig. 14.

### 5.1.2 Parameter Study

We have analyzed the effect of traction separation law on the solution of the sub-boundary value problem by considering four cases. The total reaction at the nodes on the top edge is plotted against the imposed displacement in Fig. 15a to observe the softening response of the block. In Fig. 15b, the displacement increment of an interior node is plotted against the imposed displacement on the top edge to observe the singularity of stiffness matrix $\bar{K}_{ii}$ in (57).

**CASE I: Not fully softened with linear jumps (NFS-LJ)**

The traction separation law (87) is employed with the linear interpolation of jumps in normal and tangential directions (see Eq. 35). It can be observed from the corresponding plot in Fig. 15a that the reaction at top edge first increases up to a point where the strong discontinuities suddenly develop in all the sub-elements starting with elements 1, 4 and 7. The gradual curvilinear decline in the total reaction at the top edge after $\approx 0.01$ mm of imposed displacement represents the softening behavior and locking free characteristic. Also the interior nodal increments as seen in Fig. 15b indicate that the stiffness matrix does not become singular.

**CASE II: Not fully softened with constant jumps (NFS-CJ)**

The traction separation law (87) is now employed with the constant interpolation of jumps in normal and tangential directions. From Fig. 15a the linear decline in the total reaction at the top edge after the discontinuities develop through all elements represents the softening behavior and a locking free characteristic. Also the interior nodal increments as seen in Fig. 15b represent that the stiffness matrix does not become singular.

**CASE III: Fully softened with linear jumps (FS-LJ)**

No traction separation law is employed here i.e. $t_{Γ_n}, t_{Γ_m}$ in (87) become zero as soon as the discontinuity develops. The jumps in the normal and tangential directions are linearly interpolated. In this case, the discontinuities again start with elements 1, 4 and 7 but only the top and bottom discontinuity surfaces propagate (i.e. from element 1 $→$ 2 $→$ 3 and from element 7 $→$ 8 $→$ 9) resulting in the instant softening (zero reactions at top) as seen in the corresponding plot of Fig. 15a at $\approx 0.01$ mm. Discontinuity develops through elements 5 $→$ 6 at $\approx 0.1$ mm and one can clearly see the singularity of the stiffness matrix from the infinite increments as seen in the corresponding plot of Fig. 15b resulting in the
garbage value of reactions at the top in Fig. 15a.
It is understood to be a case of a rigid body motion of element 5 as no tractions are considered to be acting along the top and bottom discontinuity surfaces and linear interpolation of discontinuity jumps results in no spurious transfer of stress (Linder and Armero, 2007) i.e. no stress locking which implies zero reactions at the interior nodes. However the discontinuity still develops after some time due to stresses in neighboring element 6.

CASE IV: Fully softened with constant jumps (FS-CJ)
Again no traction separation law is employed here with the constant interpolation of jumps in the normal and tangential directions. At \( \approx 0.01 \text{ mm} \) of imposed displacement, the discontinuity develops from element 1 \( \rightarrow \) 2 \( \rightarrow \) 3 and from element 7 \( \rightarrow \) 8 \( \rightarrow \) 9 same as in the case III resulting in the instant softening behavior of the block as seen in Fig. 15a. However, the constant interpolation of jumps leads to spurious transfer of stresses along the discontinuity which results into development of discontinuity through elements 5 \( \rightarrow \) 6 at \( \approx 0.03 \text{ mm} \) much earlier as compared to case III. From the corresponding plot in Fig. 15b, the increments of interior nodes appear to be exactly same as obtained in case II with no singularity of stiffness matrix observed.

From the analysis of above four cases, the ideal choice of parameters should be case I where a traction separation law is used with linear interpolation of jumps. The observations from case II and case IV make them equally suitable for the consideration too. However we have chosen the case IV in our representative numerical simulation shown in Sect. 5.4 merely due to the simplicity of modification of boundary conditions as discussed in chapter 4.3.5. The fully softened strong discontinuities in the sub-boundary value problem allows for the modification of deformation constraint (55) without the need of resorting to compute the contributions from displacement jumps to the displacement of boundary nodes of a sub-domain in each time step.

**Figure 15:** Fig (a) shows the total reaction at the top edge of the block and Fig (b) shows the nodal increments of an interior node both plotted against the imposed displacement on the top edge for different cases of traction separation law and jump interpolations.
5.1.3 Simulation of Bifurcation and Trifurcation

The phenomena of symmetric bifurcation and trifurcation, as discussed in chapter 3.2.2, can be precisely modeled by the multilevel embedded finite element method. Contrary to the simulations presented in Raina and Linder (2010) and Sect. 5.3 where the stress state of the branching element is poorly approximated by the linear separation mode of a strong discontinuity with an artificial constraint set for the branching angle, the simulations in Sect. 5.4 resolve the stress state of the branching element in a precise manner.

In the context of finite element simulations using multilevel EFEM, the bifurcation occurs when only two distinct strong discontinuities, belonging to the sub-elements of the global finite element, cross the boundary of global finite element as shown in the last picture of Fig. 16; whereas the trifurcation is an instance when three strong discontinuities, belonging to the sub-elements of the global finite element, cross the boundary of global finite element as shown in the last picture of Fig. 17.

We study here the stress state of a global finite element of the computational model in Sect. 5.2, where branching phenomena is taking place, by carrying out the element test as described in Sect. 5.1.1 with the original embedded finite element method (chapter 2). The $200 \times 200 \times 1\text{mm}^3$ block, representing a global finite element of Sect. 5.2, is now discretized by a patch of $5 \times 5\text{Q}_1$ elements simulating the adaptive discretization of the global finite element in the sub-boundary value problem of the multilevel embedded finite element method.

The simulation result in Fig. 17 is shown for the same linear deformation on the boundary as used in Sect. 5.1.1. Three strong discontinuities initiate from the left, with the direction given by the average of principal angles corresponding to the Gauss points of the finite element where the failure criterion is met. The finite elements lying at the front of crack tips meet failure criteria in different steps of the quasi-static loading and all three discontinuities continue propagating till the boundary on the right. This is understood
as a representation of the trifurcation phenomena in the global finite element where all three discontinuities propagate till the element edge.

In a very similar manner, the simulation results of Fig. 16 represent the bifurcation phenomena in a global finite element of the computational model in Sect. 5.2. The linear deformation state on the boundary is now modified by only changing the displacement applied on the right corner nodes of top and bottom edge to $\pm 0.02 \text{ mm}$. The three discontinuities start from the left, however the middle discontinuity surface is arrested in the middle as the failure criterion is not met in the corresponding front element as shown in the third picture of Fig. 16. The upper and lower discontinuity surfaces continue to propagate till the boundary as seen in the last picture. The simulation is stopped as soon the strong discontinuities reach the boundary of the global finite element as the linear deformation constraint applied on the boundary becomes inconsistent.

The physical aspects of crack branching outlined in Chapter 3.2.2 are precisely modeled here with the application of multilevel embedded finite element method by the accurate resolution of the stresses. This adds to the improvements observed in the qualitative features of crack branching pattern from Sect. 5.3 where the original embedded finite element method is employed to Sect. 5.4 where multilevel embedded finite element method is used.

### 5.2 Numerical Description of Experimental Setup

The experiment performed by Sharon et al. (1996) is simulated here with a reduced dimension computational model consisting of a rectangular block of height $H = 4 \text{ mm}$, width $L = 16 \text{ mm}$ with a crack of initial length $a = 2 \text{ mm}$ as shown in Fig. 18. The reduced dimension model is the same model used by Miller et al. (1999) where size effect considerations are included based upon the work of Bazant and Planas (1998). The Cartesian coordinate system is used here with the origin at the crack. The material used is PMMA and the properties used for the numerical simulations are shown in the table below.

<table>
<thead>
<tr>
<th>$E$ (GPa)</th>
<th>$\nu$</th>
<th>$t_0$ ($\mu$s)</th>
<th>$\rho$ ($\text{kg/m}^3$)</th>
<th>$c_d$ (m/s)</th>
<th>$c_R$ (m/s)</th>
<th>$f_t$ (MPa)</th>
<th>$S$ (GPa/mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.24</td>
<td>0.35</td>
<td>8</td>
<td>1190</td>
<td>2090</td>
<td>938</td>
<td>129.6</td>
<td>-24</td>
</tr>
</tbody>
</table>

The block is stretched in the $y-$ direction till a prescribed displacement $u_0$ is reached. The boundary conditions are written as

$$u_y(x, \pm H/2, t) = \begin{cases} \int v(t)dt & t \leq T \\ u_0 & t > T \end{cases} \quad \text{with} \quad v(t) = \begin{cases} v_0 t/t_0 & t \leq t_0 \\ v_0 - v_0(t - t_0)/t_d & t_0 < t < T \\ 0 & t > T \end{cases}$$

where $v_0$ is the prescribed constant velocity applied on the boundary, $t_0$ is the prescribed time during which the block is being stretched and $t_d$ is the time during which the applied velocity is ramped down to zero and $T = t_0 + t_d$. Also at the vertical boundaries ($x = 0, L$) $u_x = 0$. The loading in the form of applied velocity is shown on the left of Fig. 19 which translates into a displacement loading as shown on the right of Fig. 19. Numerical simulations are carried out for four different applied velocities of $v_0 = 6 \text{ m/s}$, $9 \text{ m/s}$, $12 \text{ m/s}$ and $15 \text{ m/s}$ with ramp down time of $t_d = 0.01 \mu$s used is all simulations.
A cohesive damage law of type (87) is used for the global finite elements, with the linear interpolation of jumps adopted in Sect. 5.3 and a constant interpolation of jumps employed in Sect. 5.4, for the values of tensile strength $f_t$ and softening modulus $S$ shown in the table above. The fracture energy of $G_f = 0.35 \text{ kJ/m}^2$ is recovered by using the above separation law which is same as used in Xu and Needleman (1994) and Miller et al. (1999). As discussed in chapter 3.2.1, the fracture energy is a function of crack velocity which is neglected in the computational model presented here.

In Sect. 5.3, the classical embedded finite element method is used to perform the numerical simulations which are later compared with the results of multilevel embedded finite element method in Sect. 5.4.

### 5.3 Simulation of Crack Branching with EFEM

The embedded finite element method presented in chapter 2 is used to perform the simulations in this section. The rectangular block is discretized into $120 \times 31$ Q1 finite elements which is considered as a suitably fine mesh in this case. The pre-existing notch of length $a$ is assumed as traction free i.e. $t_{\Gamma_n} = t_{\Gamma_m} = 0$ and the strain energy is supplied to the block up to time $t_0$ with different loading velocities. The crack starts to propagate horizontally towards the right from the tip of the notch as soon as the failure criterion is met which in our simulations is checked by the maximum principal stress of the finite element. The crack velocity $c$ increases with time as shown in Fig. 21 and at some point of time, it is seen that the crack velocity surpasses the critical velocity $c_{\text{crit}}$, which is a fraction $r$ of the Rayleigh wave speed $c_R$ such that $r = c_{\text{crit}}/c_R$. See Raina and Linder (2010) for more details on this approach.

In Sect. 5.3.2, the influence of the choice of critical velocity criterion will be presented and in Sect. 5.3.3, the influence of loading velocity $v_0$ on the crack propagation behavior will be shown.

### 5.3.1 Treatment of Branching Element

As seen in chapter 3.3.1, various numerical methodologies exist to model crack branching. One of the simplest approaches, based on embedded finite element method, is to model the branching finite element with a single strong discontinuity based on the linear separation
mode as shown on the right of Fig. 3 and also adopted in Raina and Linder (2010).

The branching angle is set a priori, so that each of the branching cracks develop in the different neighboring element to avoid multiple strong discontinuities in a single element, which is consequently corrected as the branches grow longer as observed in Fig. 22.

5.3.2 Influence of Critical Velocity Criterion

The various branching criteria were outlined in chapter 3.2.3. The simplicity of critical velocity based branching criterion makes it an attractive option to be implemented in the numerical setting. However, it is desirable to have the crack branching detected by the underlying methodology itself without having to recourse to any branching criterion.

In the numerical simulations presented here, we have chosen two different critical velocities to study their effect on the final failure pattern (micro- and macrobranching). With the increasing strain energy of the block up to \( t_0 \), the velocity of the main crack also increases due to increase of energy flux to the crack tip (See Eq. 49). First, a critical velocity of \( c_{\text{crit}} = 0.6 \times 938 = 562.8 \, \text{m/s} \) is chosen for the applied loading velocity of \( v_0 = 15 \, \text{m/s} \). The resulting crack propagation behavior is shown on the left of Fig. 20. It can be seen that at first microbranches start to appear and with increasing energy flux to the crack tip some microbranches eventually develop into macrobranches. The simulation is stopped as soon as the macrobranches reach the block edges i.e. complete failure of the block.

The same failure pattern is obtained for the critical velocity of \( c_{\text{crit}} = 0.4 \times 938 = 375.2 \, \text{m/s} \) as shown on the right of Fig. 20. Identical failure pattern is understood as the development of micro- and macrobranching at the same finite elements for different critical velocities.

Sharon et al. (1996) observed the fluctuations of the main crack velocity for higher rate of loading. This was attributed to the dynamic instabilities of the fast moving cracks which are observed in the form of microbranching formation along the main crack as shown on the right of Fig. 21. Similar behavior of main crack velocity is observed in our numerical simulations as shown on the left of Fig. 21 where the crack velocity fluctuates every time the crack branching takes place. With the appearance of a new microbranch, velocity for the main crack drops as energy released from stored strain energy in the block flows into more branches. As the main crack advances further, more energy flows back to the
Representative Numerical Simulations

$r = c_{\text{crit}}/c_R = 0.6$

$r = c_{\text{crit}}/c_R = 0.4$

Figure 20: Different critical velocity criterion lead to an identical formation of micro- and macro branching. As the crack velocity increases and reaches a critical value $c_{\text{crit}}$, branching is allowed to take place. The simulation results are shown for the applied velocity of $v_0 = 15\text{ m/s}$.

Figure 21: The velocity of main crack is monitored and plotted against the time. On the left is the numerical result of the main crack obtained for the applied velocity of $v_0 = 15\text{ m/s}$ and $r = 0.6$ which is compared to the experimental result of Sharon et al. (1996) on the right. The fluctuations are observed in the velocity of the main crack every time branching takes place.

main crack and it accelerates. Depending on the frequency of microbranch occurrence and the length each microbranch runs, oscillation pattern appears repeatedly throughout the simulation. The velocity of the main crack is computed by monitoring the time taken by the main crack to propagate through a single finite element i.e. for horizontal propagation and length $dl$ of a finite element, crack velocity is computed as $c = dl/dt$ where $dt$ is the time taken by the crack tip to cross the finite element.

The above observations lead to a conclusion that the critical velocity criterion, although used to trigger branching in finite elements, is redundant and it is expected to achieve the same results when e.g. crack branching is allowed in every finite element. In that case, no criterion is required to trigger crack branching and depending upon the stress state of finite elements, branching will take place.

5.3.3 Influence of Loading Velocity

As indicated by experimental observations in Sharon et al. (1996), the increasing strain energy leads to higher crack propagation velocities which in turn results in higher energy flux at the crack tip. Here we will study the effect of rate of increase of strain energy which is a function of applied velocity $v_0$.

Higher values of loading velocity $v_0$ simulates impact loading e.g. for $v_0 = 6\text{ m/s}$, we obtain the acceleration of $a_0 = v_0/t_0 = 750 \times 10^3 \text{ m/s}^2$ of the horizontal faces of the block which is $a_0 = 1875 \times 10^3 \text{ m/s}^2$ for the applied loading velocity of $v_0 = 15\text{ m/s}$. The strain energy release rate critically depends on this parameter as shown in Fig. 22 where
phenomena of crack branching pattern is observed for different loading velocities. For the simulation result of $v_0 = 6 \text{ m/s}$, the strain energy release rate is quite low resulting into the formation of microbranches only. Some of the microbranches tend to grow longer for the loading velocity of $v_0 = 9 \text{ m/s}$ and for higher loading velocities, the strain energy release rate is high enough to transform microbranches into macrobranches much earlier.

The simulation results are in good agreement with the general observed phenomena in experimental results. The velocity of the main crack starts fluctuating as soon as it reaches a critical value which is caused by the development of microbranches and for the higher rate of external loading, some microbranches grow longer to form macrobranches. Also the macrobranches happen to appear sooner for the higher rate of loading both in experimental as well as simulation results.

It is to be noted here that the branching is allowed to take place as an instance of trifurcation (See chapter 3.2.2) i.e. the main crack continues to propagate after the branching takes place which is the result of the choice of branching element represented by the linear separation modes of a single discontinuity as discussed in Sect. 5.3.1.

### 5.4 Simulation of Crack Branching with Multilevel EFEM

The simulations in this section are carried out by using the multilevel embedded finite element method as presented in chapter 4 with the same computational model shown in Sect. 5.2 which is now discretized into $80 \times 21$ Q1 finite elements. A comparatively coarser mesh is chosen due to the higher computational cost associated with the multilevel embedded finite element method and the fact that the crack propagation is independent of the underlying discretization in case of embedded finite element method.

The same assumption of a traction free pre-existing notch of length $a$ is made and the strain energy is supplied to the block up to time $t_0$ with different loading velocities. As shown earlier, the crack starts to propagate horizontally towards the right from the tip of the notch as soon as the failure criterion is met. Again the crack velocity $c$ increases with time as shown in Fig. 24 and at some point of time, it is seen that the crack velocity surpasses the critical velocity $c_{\text{crit}}$, set as a fraction $r$, of the Rayleigh wave speed $c_R$. 
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Figure 23: For the applied velocity of \( v_0 = 15 \, \text{m/s} \), different critical velocity criterion again lead to an identical formation of micro- and macro branching by using the multilevel EFEM.

\[
\begin{align*}
\text{Relative crack velocity with respect to } & \text{cR} \\
\text{time [\mu s]} & \\
\text{Numerical result} & \\
\text{Experimental result}
\end{align*}
\]

\[
\begin{align*}
\text{Crack velocity [m/s]} & \\
\text{time [\mu s]} & \\
\text{Numerical result} & \\
\text{Experimental result}
\end{align*}
\]

Figure 24: On the left is the numerical result obtained for the applied velocity of \( v_0 = 15 \, \text{m/s} \) and \( r = 0.6 \) by using the multilevel EFEM which is compared to the experimental result of Sharon et al. (1996) on the right. Again the fluctuations in the velocity of the main crack are observed every time the branching phenomena occurs.

We will again study the influence of choice of critical velocity criterion in Sect. 5.4.1 and influence of loading velocity in Sect. 5.4.2 on our simulation results. In Sect. 5.1.3, the phenomena of bifurcation and trifurcation is simulated for the first time by using the multilevel embedded finite element method.

5.4.1 Influence of Critical Velocity Criterion

As discussed in Sect. 5.3.2, two different critical velocities, represented by \( r = 0.6 \) and \( r = 0.4 \), are chosen to study their effect on micro- and macrobranching. As soon as the principal stresses in the finite element in front of the notch tip reach a critical value, the crack starts to propagate towards the right surface. The velocity of the main crack increases with time till it reaches the set critical value where the branching is now allowed to take place.

The behavior of crack propagation in terms of micro- and macrobranching for different values of \( r \) and the applied loading velocity of \( v_0 = 15 \, \text{m/s} \) is shown in Fig. 23. The identical failure patterns are once again achieved in terms of development of micro- and macrobranches at the same finite elements for different values of critical velocities.

The velocity fluctuations associated with the fast moving crack, as shown on the left of Fig. 24 for the numerical results and observed in the experimental results as shown on the right of Fig. 24, is observed every time crack branching takes place and is computed in the same manner as mentioned before. The fluctuations in the crack velocity depend upon the frequency of occurrence of micro- and macrobranches.

Based on the observations above, the conclusion made in Sect. 5.3.2 can be reiterated here
that the critical velocity criterion used to trigger crack branching artificially is unnecessary and it is expected to achieve the same results in case no branching criterion is employed.

5.4.2 Influence of Loading Velocity

Following the discussion of Sect. 5.3.3, where it was established that the strain energy release rate increases for higher loading velocity, we study the same effect of influence of loading velocity on the crack branching pattern by using the multilevel embedded finite element method.

The simulation results obtained for the loading velocities of $v_0 = 6 \text{ m/s}$, $9 \text{ m/s}$, $12 \text{ m/s}$ and $15 \text{ m/s}$ are shown in Fig. 25 where it is observed that for a lower velocity of $v_0 = 6 \text{ m/s}$, only microbranches appear along the main crack due to lower strain energy release rate. For $v_0 = 9 \text{ m/s}$, microbranches tend to grow longer with more energy flux flowing into branching crack tips and for higher loading velocities of $12 \text{ m/s}$ and $15 \text{ m/s}$, macro-branches develop out of microbranches much earlier than before.

One of the interesting results here is the simulation of both the bifurcation and the trifurcation phenomena as observed in Fig. 25 where bifurcation is taking place at the rightmost end of the main crack tip e.g. for the simulation result of $v_0 = 12 \text{ m/s}$. Such a phenomena is a result of the solution of the sub-boundary value problem associated with each finite element where the crack branching criterion is met and will be discussed in more detail in Sect. 5.1.3.
5.5 Conclusion

In this study, a computational tool for the numerical modeling of crack branching in dynamic fracture of brittle materials is developed. The new methodology, referred to as a multilevel embedded finite element method, is formulated where a finite element where the crack branching is taking place is solved as a separate boundary value problem in the incremental time step \([t_n, t_{n+1}]\) of the transient setting employed here. A kinematic constraint is used on the boundary of this finite element which connects the nodes at different levels. The averaged properties are then computed by matching the virtual energies associated with the different levels at the local equilibrium.

The experiment in Sharon et al. (1996) is simulated in the numerical examples with the reduced dimensions. The restrictions of crack path, branching angle and number of branches associated with the numerical modeling of such a phenomena in the previous works are no longer employed in the simulations performed here with the new methodology. In addition to the computation of the averaged properties, the separate boundary value problem also determines the physical aspects of crack branching i.e. the branching angle, bifurcation or trifurcation of the main discontinuity surface. Thus, the precise state of the finite element where branching is taking place is computed with the new methodology.
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